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Bergshoeff, E.; Roo, M. de; Wit, B. de

Published in:
 Nuclear Physics B

DOI:
[10.1016/0550-3213\(83\)90159-1](https://doi.org/10.1016/0550-3213(83)90159-1)

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Document Version
 Publisher's PDF, also known as Version of record

Publication date:
 1983

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Bergshoeff, E., Roo, M. D., & Wit, B. D. (1983). Conformal supergravity in ten dimensions. *Nuclear Physics B*, 217(2). [https://doi.org/10.1016/0550-3213\(83\)90159-1](https://doi.org/10.1016/0550-3213(83)90159-1)

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CONFORMAL SUPERGRAVITY IN TEN DIMENSIONS

E. BERGSHOEFF

Instituut-Lorentz, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands

M. DE ROO

Instituut voor Theoretische Natuurkunde, P.O. Box 800, 9700 AV Groningen, The Netherlands

B. DE WIT

NIKHEF-H, P.O. Box 41882, 1009 DB Amsterdam, The Netherlands

Received 6 December 1982

We present the complete off-shell structure of conformal supergravity in ten dimensions. It is based on $128 + 128$ degrees of freedom and its formulation requires differential constraints. We study how these constraints are resolved in four and five dimensions. Covariant conditions are given that restrict conformal supergravity to its on-shell Poincaré counterpart. In ten dimensions the relationship between the two theories has new and unusual aspects, which we explore in a variety of ways. We rewrite on-shell Poincaré supergravity in a superconformally invariant form, from which we deduce that its off-shell version must contain at least a scalar (chiral) multiplet. We analyze some aspects of the non-linear structure of the field representation based on the conformal fields combined with one scalar multiplet.

1. Introduction and summary

Conformal invariance is the highest degree of space-time symmetry that a field theory without dimensional parameters can have. The superconformal transformations are the supersymmetric generalization of this symmetry. In four dimensions they consist of the conformal transformations and two different kinds of supersymmetries, called Q and S supersymmetry. In extended supersymmetry one has N independent invariances of each type; in addition, a closed algebraic structure requires to include chiral $(S)U(N)$ transformations as well. In conformal (or Weyl) supergravity the role that these symmetries play is that they allow the definition of an off-shell *irreducible* field representation containing the spin-2 gravitational fields. This multiplet of fields is called the Weyl multiplet; it contains the gauge fields of the superconformal symmetries and forms a representation of the superconformal algebra. For $N \leq 4$ these multiplets have been presented in [1–4].

The $N = 1$ and $N = 2$ Weyl multiplets in four dimensions have been useful in clarifying the off-shell structure of theories possessing a lesser degree of gauge invariance, such as $N = 1$ and $N = 2$ Poincaré supergravity [5, 6]. The reason is

that one can view the off-shell Poincaré supergravity fields as constructed out of the Weyl multiplet and a number of so-called compensating supermultiplets. Inequivalent field representations of Poincaré supergravity differ in the choice of the compensating supermultiplets, but always contain the Weyl multiplet. Although the resulting field representation is thus reducible with respect to the superconformal symmetries, it is gauge equivalent to an irreducible field representation of the super-Poincaré symmetries. To see that this is indeed the case, it is most convenient to impose a number of gauge conditions. Those conditions fix the values of the compensating fields, thereby directly reducing the gauge invariances to those of the super-Poincaré theory. This construction of Poincaré theories in a superconformal framework requires knowledge of a variety of superconformal multiplets, which can be used to provide the necessary compensating fields when coupled to conformal supergravity. For $N = 4$ not much is known about such multiplets and therefore the compensating mechanism has not been applied in this case, but results have been given in five dimensions [7].

In this paper we consider an implementation of superconformal ideas in the context of supergravity in ten dimensions. In particular, we present the full non-linear structure of conformal supergravity, and study its relation with Poincaré supergravity. Our hope is that by studying conformal supergravity in another context than the familiar four-dimensional one we may learn something about the way in which the conformal theories can be realized for $N > 4$. The construction of the Weyl multiplet is based on an analysis of the supermultiplet of currents which describe the coupling of supersymmetric matter to supergravity. The reason for going to ten dimensions is that supersymmetric matter exists only for $d \leq 10$ (or equivalently $d = 4$, $N \leq 4$) so that this is the highest-dimensional supergravity theory that can be constructed in this way.

The linearized transformation rules of the $d = 10$ Weyl multiplet were already found in [8] from an analysis of the $d = 10$ Maxwell supercurrent. This supercurrent is reducible; it contains a submultiplet of $128 + 128$ components, whereas the remaining degrees of freedom form a constrained scalar (chiral) superfield. In the non-abelian case the scalar superfield part of the current is unconstrained [9]. The $128 + 128$ current submultiplet is associated with the fields of conformal supergravity, because it is the smallest off-shell multiplet that contains the energy-momentum tensor. Indeed, these currents satisfy the constraints that are appropriate for currents that couple to superconformal gauge fields. However, a non-trivial aspect is that the decomposition of the $d = 10$ supercurrent into its two submultiplets is realized in a non-local way. As a consequence the linear transformation rules of the $d = 10$ Weyl multiplet contain non-local terms. In order to construct the complete non-linear theory one first has to decide how to deal with this complication. In this paper we show how to avoid the non-local character of the transformations by introducing new fields which are subject to differential constraints. Hence these fields do not represent new degrees of freedom. We call them would-be compensat-

ing fields, because they exhibit transformations that are typical for compensating fields; however, because of the constraints their usefulness is rather limited.

The presence of the differential constraints is a departure from the four-dimensional situation. In four dimensions the superconformal gauge transformations alone are sufficient to restrict the fields to the irreducible Weyl multiplet. For reasons that we do not systematically understand these transformations no longer suffice in a higher dimensional context and one is forced to introduce differential constraints as well. To exhibit the difference with the conventional situation one may reduce the $d = 10$ Weyl multiplet to lower space-time dimensions. In this paper we shall demonstrate how the differential constraints on the would-be compensators are resolved in the reduction to four and five dimensions. In both cases the resulting $N = 4$ conformal supergravity theory is based on $128 + 128$ degrees of freedom, and the constraints are avoided by absorbing the differential operators into the definition of the various fields. Another crucial step in the elimination of the constraints is provided by the introduction of new internal gauge transformations corresponding to $\text{Sp}(4)$ or $\text{SU}(4)$ for $d = 5$ or 4 , respectively. The five-dimensional conformal supergravity theory has not been given before. It is straightforward to show that this theory is, in fact, gauge equivalent to the supergravity theory proposed in [10].

Our construction of the Weyl multiplet emphasizes irreducibility of the field representation rather than its possible origin as a gauge theory based on a superalgebra of abstract generators. We should point out that also in $d = 4$ the relation between the superconformal algebras and conformal supergravity is not very direct (for a review, see [11]). Only for $N = 1$ the gauge fields themselves correspond directly to the Weyl multiplet, but if $N > 1$ one has to introduce additional matter fields which bear no direct relationship to the underlying generator algebra. Recently a $d = 10$ superconformal algebra has been considered [12]; besides the conventional superconformal generators this algebra contains many bosonic generators which are absent in our formulation. It is at present unclear whether there exists a relation between this algebra and our results, but if there is one, it may shed some light on the differential constraints.

Although the presence of the differential constraints presents an obstacle for a straightforward application of the compensating mechanism, it does not prevent us from constructing invariants, either for conformal supergravity itself, or for conformal supergravity coupled to matter. For instance, we discuss the full non-linear coupling of conformal supergravity to $d = 10$ supersymmetric Yang–Mills theory [13]. It turns out that its fields indeed define an (on-shell) representation of the full algebra of superconformal gauge transformations. We compare this result to previous on-shell constructions of Einstein–Yang–Mills supergravity [14, 15]. This can be done because there is an intrinsic relation between the Weyl multiplet and on-shell Poincaré supergravity. We shall present a set of covariant conditions that restrict the Weyl multiplet accordingly. These conditions are not the ones that

one might expect from experience in four dimensions, and this clarifies the apparent discrepancy noted before [7, 10, 15] that the supercurrent in higher dimensions does not seem to contain the components that are required for a coupling to all the physical Poincaré supergravity fields. Because the Weyl multiplet represents the on-shell structure of Poincaré supergravity it can be used in the discussion of on-shell invariant counterterms, precisely as in four dimensions [16]. Nevertheless the relationship between the superconformal and the super-Poincaré theory remains considerably more subtle in ten dimensions. For instance, if we describe the on-shell Poincaré theory in terms of a six-rank antisymmetric gauge field [14] rather than the more conventional two-rank field [15, 17], this theory is described in terms of the same fields as (off-shell) conformal supergravity. Somewhat surprisingly, it is also possible to write the Q -supersymmetry transformations of the superconformal multiplet in such a way that they take on the same form as the Poincaré supersymmetry transformations. In this paper we shall discuss these aspects of the relationship between Poincaré and Weyl supergravity in detail.

It is possible to exploit this relationship in the search for possible off-shell versions of Poincaré supergravity. By introducing two compensating fields it is generally possible to rewrite the known Poincaré supergravity lagrangian in such a way that it becomes invariant under local dilatations and S supersymmetry. The superconformal transformations of the compensating fields can be deduced from the supersymmetry transformations of the Poincaré fields combined with the knowledge of conformal supergravity and the relation between the two theories. It turns out this new form of the lagrangian reveals a structure where the compensating fields multiply the differential constraint equations of the superconformal theory as Lagrange multipliers. One can also show that these Lagrange multipliers must be contained in a scalar (chiral) multiplet. Consequently the same must be true for the constraints, in the sense that at least a subset of them must transform as the components of a scalar multiplet. By introducing new degrees of freedom corresponding to the constraints one may now obtain an unconstrained field representation consisting of the $128 + 128$ components of the Weyl multiplet with an extra submultiplet. The latter should be viewed as a supersymmetric generalization of the differential constraints, which when put to zero restrict the fields to conformal supergravity. In this way we have relaxed the requirement of irreducibility in a way consistent with supersymmetry which brings us outside the context of pure conformal supergravity.

The above arguments show that an off-shell field representation for $d = 10$ Poincaré supergravity in terms of which one may formulate an invariant action should be based on a multiplet that contains at least a scalar submultiplet besides the superconformal fields. Therefore, the *minimal* field representation contains precisely the fields that couple to the Yang–Mills supercurrent [8, 9]. In addition we find at least one independent scalar multiplet of Lagrange multipliers. In principle it is straightforward to avoid the differential constraints by introducing new degrees

of freedom. After ignoring the constraints one adds new fields in the transformation laws of the superconformal fields whose variations are then required to reestablish the closure of the superconformal algebra. Subsequently the results may be completed by iteration. It is not obvious that such a program will be successful for the full non-linear theory, although there are no conceivable problems at the linearized level. A crucial point is that the original commutation relations of the superconformal algebra will be modified by terms that contain the new fields. We investigate some of the non-linear aspects of the field representation with only a scalar submultiplet. We find a number of non-linear modifications associated with this submultiplet, and at this stage a completion along these lines seems perfectly possible. These results are in fact relevant for the recently obtained off-shell formulation of linearized $d = 10$ Poincaré supergravity [9].

Based on the above arguments it is possible to obtain a complete superconformally invariant version of the Poincaré action; after imposing the appropriate superconformal gauge conditions one obtains a Poincaré supergravity action with its auxiliary fields. For the minimal extension of the field representation the results are expected to coincide with those of [9]. This is confirmed by a calculation of some of the new terms in the action, which has many of the same ingredients, although the superconformal scheme leads to a different arrangement of terms.

The article is organized as follows. In sect. 2 we discuss conformal gravity in d dimensions. We explain how to construct conformally invariant actions by means of would-be compensators which are subject to differential constraints. In sect. 3 we present the complete non-linear multiplet of ten-dimensional conformal supergravity. Furthermore we construct a linearized superconformal action for the Weyl multiplet. In sect. 4 we give the reduction to lower space-time dimensions. Section 5 is devoted to a discussion of various aspects of the relation between Poincaré and conformal supergravity in ten dimensions. The coupling of conformal supergravity to the supersymmetric Yang–Mills theory is presented and compared to Einstein–Yang–Mills supergravity. In sect. 6 we discuss a superconformal formulation of Poincaré supergravity, which brings us outside the context of pure conformal supergravity. Some technical details are relegated to two appendices. Throughout this paper we employ the notation of [15].

2. Conformal gravity in d dimensions

It is possible to view conformal gravity as the gauge theory of $\text{SO}(d, 2)$, the group of conformal transformations in d space-time dimensions. The $\frac{1}{2}(d+2)(d+1)$ generators of these transformations correspond to d infinitesimal translations (P_a), $\frac{1}{2}d(d-1)$ infinitesimal Lorentz transformations (M_{ab}), d infinitesimal conformal boosts (K_a) and the infinitesimal dilatation (D), to which one assigns the gauge fields e_μ^a , ω_μ^{ab} , f_μ^a and b_μ . Their transformations follow from $\text{SO}(d, 2)$, and take

the form

$$\begin{aligned}
 \delta e_\mu^a &= \mathcal{D}_\mu \Lambda_P^a - \Lambda_D e_\mu^a, \\
 \delta \omega_\mu^{ab} &= 2\Lambda_K^{[a} e_\mu^{b]}, \\
 \delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_\mu^a, \\
 \delta f_\mu^a &= \mathcal{D}_\mu \Lambda_K^a + \Lambda_D f_\mu^a,
 \end{aligned} \tag{2.1}$$

where Λ_P^a , Λ_K^a and Λ_D denote the parameters of P , K and D transformations. The Lorentz transformations follow directly from the assignment of the Lorentz indices $a, b = 1, 2, \dots, d$; the derivatives \mathcal{D}_μ are covariant with respect to Lorentz (M) and scale transformations (D). Covariant curvature tensors for these fields take the following form:

$$\begin{aligned}
 R_{\mu\nu}^a(P) &= \mathcal{D}_{[\mu} e_{\nu]}^a, \\
 R_{\mu\nu}^{ab}(M) &= \partial_{[\mu} \omega_{\nu]}^{ab} - \omega_{[\mu}^{ac} \omega_{\nu]}^{cb} - 2f_{[\mu}^{[a} e_{\nu]}^{b]}, \\
 R_{\mu\nu}(D) &= \partial_{[\mu} b_{\nu]} - f_{[\mu}^a e_{\nu]}^a, \\
 R_{\mu\nu}^a(K) &= \mathcal{D}_{[\mu} f_{\nu]}^a,
 \end{aligned} \tag{2.2}$$

and satisfy the standard Bianchi identities.

This gauge theory is thus based on $\frac{1}{2}(d+2)(d+1)(d-1)$ field degrees of freedom. However, its $SO(d, 2)$ gauge group is as yet unrelated to reparametrizations of space-time, governed by independent general coordinate transformations which transform gauge fields as covariant vectors. In order to convert this theory into a gauge theory of space-time transformations, one imposes a set of conventional constraints that algebraically express some of the gauge fields in terms of the others. There is a unique set of curvature constraints of this type which preserves all $SO(d, 2)$ transformations except P :

$$\begin{aligned}
 R_{\mu\nu}^a(P) &= 0, \\
 R_{\mu\nu}^{ab}(M) e_\nu^b &= 0,
 \end{aligned} \tag{2.3}$$

where e^μ_a is the inverse of e_μ^a . The first constraint expresses ω_μ^{ab} in terms of e_μ^a and b_μ and the second one restricts f_μ^a in a similar way. Although ω_μ^{ab} and f_μ^a are no longer independent their M , K and D transformations remain the same, since the constraints are invariant under those symmetries. The P transformation rules change, however, but it is simple to show that those variations are no longer independent, and can be uniformly expressed as a general coordinate transformation augmented by field-dependent M , K , and D transformations. Therefore, P transformations can be discarded, and we have now obtained a gauge theory of space-time transformations in which e_μ^a is identified as the (inverse) d -bein field. The dilatation gauge field b_μ remains as a second independent field. Since b_μ is the only independent

field transforming under K , the K transformations of the dependent fields ω_μ^{ab} and f_μ^a are thus generated by the variation of b_μ in the corresponding expressions.

Combining the constraints (2.3) with the Bianchi identities on the curvatures (2.2) leads to the equations

$$\begin{aligned} R_{\mu\nu}(D) &= 0, \\ R_{[\mu\nu}{}^{ab}(M)e_{\rho]}{}^b &= 0, \\ \mathcal{D}_{[\rho}R_{\mu\nu]}{}^{ab}(M) - 2e_{[\rho}{}^{[a}R_{\mu\nu]}{}^{b]}(K) &= 0. \end{aligned} \quad (2.4)$$

This shows that the remaining $\frac{1}{2}(d+1)(d-2)$ field degrees of freedom can be expressed entirely in terms of the curvature tensor $R_{\mu\nu}{}^{ab}(M)$. We have thus obtained conformal gravity as a theory of a pure spin 2; indeed massive spin-2 states have $\frac{1}{2}(d+1)(d-2)$ helicity components in d dimensions.

The fact that the degrees of freedom coincide with those of a *massive* spin-2 particle confirms that we have not yet imposed equations of motion which would restrict the fields to take values on the light cone. The restriction of the fields to the irreducible representation of highest spin is a characteristic feature of conformal gravity, which is achieved by introducing all the gauge invariances of the conformal group. Actually, this is analogous to what happens for standard gauge theories of the Yang–Mills type, where the gauge fields are restricted to $d-1$ degrees of freedom corresponding to the massive spin-1 representation of the Poincaré group; in that case gauge invariance removes one degree of freedom of the vector potential. Hence extra gauge invariance can be used to restrict a field representation to a higher degree of irreducibility, although as we shall see in this paper it is not always sufficient for maximal irreducibility.

It is possible to couple conformal gravity to matter multiplets in the standard way. An important role is played by derivatives that are covariant with respect to M , K and D transformations. A single covariant derivative for a K -inert matter field is easy to construct. For instance, for a scalar ϕ or a spinor χ with Weyl weight w , we have

$$\begin{aligned} D_\mu\phi &= \mathcal{D}_\mu\phi = (\partial_\mu - wb_\mu)\phi, \\ D_\mu\chi &= \mathcal{D}_\mu\chi = (\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab} - wb_\mu)\chi. \end{aligned} \quad (2.5)$$

However, to construct a second-order derivative one must also consider the K transformations of $D_\mu\phi$ and $D_\mu\chi$, generated by their dependence on ω_μ^{ab} and b_μ . It is straightforward to find

$$\begin{aligned} \delta_K(D_\alpha\phi) &= -w\Lambda_K{}^\alpha\phi, \\ \delta_K(D_\alpha\chi) &= (\frac{1}{2}\Gamma_a\Gamma_b - (w + \frac{1}{2})\delta_{ab})\chi\Lambda_K{}^b. \end{aligned} \quad (2.6)$$

Therefore, we have

$$D_\mu D_\alpha\phi = (\partial_\mu - (w+1)b_\mu)D_\alpha\phi - \omega_\mu{}^{ab}D_b\phi + wf_\mu{}^a\phi. \quad (2.7)$$

To construct invariant actions for ϕ and χ , one considers the K variation of the conformal d'Alembertian and the Dirac operator

$$\begin{aligned}\delta_K(D^a D_a \phi) &= (d - 2 - 2w) \Lambda_K^a D_a \phi, \\ \delta_K(\not{D}\chi) &= (\tfrac{1}{2}d - \tfrac{1}{2} - w) \Lambda_K^a \Gamma_a \chi.\end{aligned}\quad (2.8)$$

Hence for a scalar field with $w = \frac{1}{2}(d - 2)$ the d'Alembertian is K invariant, and the lagrangian

$$\mathcal{L} \propto e \phi \square^c \phi \quad (2.9)$$

leads to a conformally invariant action; for a spinor with $w = \frac{1}{2}(d - 1)$ the conformally invariant action follows from

$$\mathcal{L} \propto e \bar{\chi} \not{D} \chi. \quad (2.10)$$

The high degree of gauge invariance of conformal gravity seems to restrict the variety of invariant actions. However, this is not quite so, because one may introduce extra degrees of freedom in the form of matter fields. Such fields, which are called compensating fields, enable one to write actions in a conformally invariant way that are gauge equivalent to actions that are usually not considered to be of the conformal type. For instance, the action corresponding to (2.9) is gauge equivalent to Einstein gravity in d dimensions, as one can show directly by imposing a set of gauge conditions that break D and K symmetry,

$$b_\mu = 0, \quad \phi = \kappa^{-1}, \quad (2.11)$$

and by realizing that the term $f_\mu{}^\mu \phi^2$ in (2.9) becomes the standard Ricci scalar. This follows from the second constraint (2.3) which implies

$$f_\mu{}^a = \frac{2}{d-2} R_{\mu a}(e) - \frac{1}{(d-1)(d-2)} e_\mu{}^a R(e), \quad (2.12)$$

where we have already suppressed the dependence on b_μ according to (2.11). The dimensionful constant κ , which is related to Newton's constant, is put equal to one henceforth.

The compensating fields also play a role when coupling matter to conformal gravity for matter actions which are not manifestly scale invariant. For example, Maxwell's theory in d dimensions has a conformally invariant action corresponding to

$$\mathcal{L} = -\frac{1}{4} e \phi^{2(d-4)/(d-2)} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2, \quad (2.13)$$

where ϕ is the compensating scalar.

There is an alternative way for constructing a large variety of actions without the need for introducing new degrees of freedom, which entails differential constraints. This can be done by means of a would-be compensator which is subject

to a constraint. As we shall see in the next section such a construction arises naturally in the context of $d = 10$ conformal supergravity. As an illustration consider a scalar field ϕ with $w = \frac{1}{2}(d-2)$, which is restricted by an invariant condition

$$\square^c \phi = 0. \quad (2.14)$$

It is important to understand that (2.14) does not imply that ϕ is a massless field, but instead it restricts part of the K gauge field according to

$$f_\mu{}^\mu = -\frac{2}{d-2} \phi^{-1} \mathcal{D}^a \mathcal{D}_a \phi. \quad (2.15)$$

On the other hand, $f_\mu{}^\mu$ is already restricted by the conventional constraints (2.3), which leads to (2.12). Therefore ϕ is related to the d -bein fields by a differential constraint. It can be shown that the constraints (2.3) together with (2.14) describe the $\frac{1}{2}(d+1)(d-2)$ states of massive spin 2.

One can now construct conformally invariant actions by using ϕ as a compensating field, disregarding the fact that ϕ is restricted. An example of a conformally invariant lagrangian is

$$\mathcal{L} \propto e e_{[a}{}^\mu e_b{}^\nu e_c{}^\rho e_d{}^\sigma \phi^{2(d-4)/(d-2)} R_{\mu\nu}{}^{ab}(M) R_{\rho\sigma}{}^{cd}(M). \quad (2.16)$$

This is the generalization of the $d = 4$ conformal gravity action to d dimensions. In $d = 4$ the dependence on ϕ disappears, so that the differential constraint can be ignored. The lagrangian (2.16) can also be expressed in terms of a redefined d -bein field which is inert under dilatations

$$(e_\mu{}^a)^{\text{new}} = \phi^{2/(d-2)} e_\mu{}^a. \quad (2.17)$$

In this case (2.16) no longer depends on ϕ . However, (2.14) then takes on the form

$$R(e^{\text{new}}) = 0. \quad (2.18)$$

On using the conventional constraint (2.3) (or by substituting the solution of the $f_\mu{}^a$ equation of motion) (2.16) becomes

$$\mathcal{L} \propto -\frac{8}{3} \frac{d-3}{d-2} e \left[R_{\mu\nu}(e) R^{\mu\nu}(e) - \frac{d}{4(d-1)} R^2(e) \right] \phi^{2(d-4)/(d-2)}. \quad (2.19)$$

A field ϕ that satisfies (2.14) cannot be used to construct a conformal invariant which is gauge equivalent to Einstein gravity. For that purpose one still needs an independent compensating field. Its action which we have given in (2.9) does not contain the would-be compensating field so that the constraint (2.14) does not play a role.

3. Conformal supergravity in ten dimensions

In this section we present the complete non-linear multiplet of ten-dimensional conformal supergravity. The starting point for its construction is the off-shell multiplet of fields which is conjugate to the Maxwell supercurrent. Let us therefore briefly review the structure of the supercurrent multiplet.

The $d = 10$ Maxwell theory [13] is described by the lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\chi}\not{\partial}\chi, \quad (3.1)$$

where χ is a chiral Majorana spinor; (3.1) is invariant under the supersymmetry transformations

$$\delta A_\mu = \frac{1}{2}\bar{\epsilon}\Gamma_\mu\chi, \quad \delta\chi = -\frac{1}{4}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon. \quad (3.2)$$

The energy-momentum tensor and the supersymmetry current are given by

$$\theta_{\mu\nu} = 4F_{\mu\alpha}F^\alpha_\nu - \delta_{\mu\nu}F^2 + \bar{\chi}(\Gamma_\mu\partial_\nu + \Gamma_\nu\partial_\mu)\chi, \quad (3.3)$$

$$J_\mu = \frac{1}{4}\Gamma \cdot F\Gamma_\mu\chi, \quad (3.4)$$

which are conserved but not (gamma-)traceless. The components of the supercurrent are found by considering successive variations of J_μ , using (3.2) and the Maxwell field equations (the variation of $\theta_{\mu\nu}$ contains only J_μ itself). The complete structure of the supercurrent was analyzed in ref. [8]. It was found that this off-shell multiplet consists of $5760 + 5760$ bosonic + fermionic components, but that it is reducible and contains a submultiplet with only $128 + 128 = 2^8$ components. The remaining $5632 + 5632 = 44 \times 2^8$ degrees of freedom form a constrained scalar (chiral) superfield. In the non-abelian case [9] the scalar superfield part of the current is unconstrained, and the multiplet then contains all 2^{16} components of the scalar superfield, plus of course the 2^8 degrees of freedom of the submultiplet.

For our present purposes it is sufficient to study the variation of J_μ . One finds [15]

$$\delta J_\mu = -\frac{1}{8}\Gamma_\nu\epsilon\theta_{\mu\nu} - \frac{1}{8}\Gamma_\mu^{\lambda_1\cdots\lambda_4}\epsilon W_{\lambda_1\cdots\lambda_4} + \frac{1}{96}\Gamma_{\mu\nu}\partial_\nu\Gamma^{\rho\sigma\tau}\epsilon X_{\rho\sigma\tau}. \quad (3.5)$$

Here we have defined

$$W_{\mu\nu\rho\sigma} = \frac{1}{2}F_{[\mu\nu}F_{\rho\sigma]} + \frac{1}{12}\partial_{[\mu}X_{\nu\rho\sigma]}, \quad (3.6)$$

$$X_{\mu\nu\rho} = \bar{\chi}\Gamma_{\mu\nu\rho}\chi. \quad (3.7)$$

The variations of $\theta_{\mu\nu}$ and $W_{\mu\nu\rho\sigma}$ equal

$$\delta\theta_{\mu\nu} = 2\bar{\epsilon}\Gamma_{(\mu\lambda}\partial_{\lambda}J_{\nu)},$$

$$\delta W_{\mu\nu\rho\sigma} = \frac{1}{2}\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\rho}(J_{\sigma]} - \frac{1}{9}\Gamma_{\sigma]}\Gamma \cdot J). \quad (3.8)$$

Therefore the currents $\theta_{\mu\nu}$, J_μ and $W_{\mu\nu\rho\sigma}$ would form a multiplet, were it not for the variation of J_μ into $X_{\mu\nu\rho}$. The number of degrees of freedom at this stage is 129 bosonic ($\theta_{\mu\nu}$, $W_{\mu\nu\rho\sigma}$) and 144 fermionic (J_μ).

To obtain a supermultiplet based on these currents one should somehow suppress the third term in the variation (3.5). This can be done by introducing a gauge transformation which acts on J_μ according to

$$\delta J_\mu = \Gamma_{\mu\nu} \partial_\nu \xi(x), \quad (3.9)$$

where $\xi(x)$ is an arbitrary space-time dependent spinorial parameter. This transformation reduces the number of fermionic degrees of freedom to 128. However, supersymmetry does not commute with (3.9); this leads to a second gauge transformation which acts on $\theta_{\mu\nu}$:

$$\delta \theta_{\mu\nu}(x) = (\delta_{\mu\nu} \square - \partial_\mu \partial_\nu) \Lambda(x), \quad (3.10)$$

with $\Lambda(x)$ a space-time dependent scalar parameter. The number of bosonic degrees of freedom is now also reduced to 128, and one has indeed obtained a full supermultiplet.

At this point one may proceed in two different ways. Either one defines the multiplet in terms of a subset of current components that are inert under the transformations (3.9) and (3.10), or one directly constructs a supermultiplet based on the full currents. In the latter approach the gauge fields that couple to these currents must be subject to constraints induced by the need to keep this coupling invariant under (3.9) and (3.10). These constraints are (linearized)

$$\begin{aligned} \Gamma^{\mu\nu} \partial_\mu \psi_\nu &= 0, \\ \square h_{\mu\mu} - \partial_\mu \partial_\nu h_{\mu\nu} &= 0, \end{aligned} \quad (3.11)$$

where ψ_μ is the gravitino field and $h_{\mu\nu}$ represents the deviation of the symmetric part of the 10-bein field from its flat-space value. Hence we expect that the fields of ten-dimensional conformal supergravity are subject to differential constraints, a property that we have already alluded to in the previous section.

For the moment we will follow the first approach, which has been given in [8], but it turns out that the formulation of the theory that we will obtain encompasses both options in a natural way. The currents invariant under (3.9) and (3.10) are precisely the (gamma-)traceless parts of J_μ and $\theta_{\mu\nu}$, together with the full $W_{\mu\nu\rho\sigma}$. This signals the fact that now the gauge fields will acquire extra gauge transformations, namely dilatations and S supersymmetry (linearized)

$$\begin{aligned} \delta h_{\mu\nu}(x) &= -\delta_{\mu\nu} A_D(x), \\ \delta \psi_\mu(x) &= -\Gamma_\mu \eta(x). \end{aligned} \quad (3.12)$$

The fields and their linearized supersymmetry transformations take the following form:

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta \psi_\mu &= \mathcal{D}_\mu(\omega(e)) \epsilon + \Gamma_\mu^{\lambda_1 \dots \lambda_7} \epsilon \partial_{\lambda_1} A_{\lambda_2 \dots \lambda_7}, \end{aligned}$$

$$\delta A_{\lambda_1 \dots \lambda_6} = \frac{1}{960} \left\{ \bar{\epsilon} \Gamma_{[\lambda_1 \dots \lambda_5} \psi_{\lambda_6]} - \frac{1}{9} \bar{\epsilon} \Gamma_{\lambda_1 \dots \lambda_6} \not{\square} \not{\partial} \Gamma^{\mu\nu} \partial_\mu \psi_\nu \right\}. \quad (3.13)$$

The non-locality in $\delta A_{\lambda_1 \dots \lambda_6}$ is not surprising and is due to the fact that we had to project out the traceless parts of J_μ and $\theta_{\mu\nu}$. Compared to the original formulation of this multiplet [8] we find it advantageous to use a 6-index rather than a 4-index tensor gauge field; this tensor gauge field has corresponding gauge transformations

$$\delta A_{\lambda_1 \dots \lambda_6} = \partial_{[\lambda_1} \xi_{\lambda_2 \dots \lambda_6]}. \quad (3.14)$$

Before we proceed from (3.13) to the complete non-linear theory, we must first decide how to deal with the non-local term in δA . It occurs in the combination

$$\lambda = -\frac{1}{9} \not{\square} \Gamma^{\mu\nu} \partial_\mu \psi_\nu, \quad (3.15)$$

which suggests that we introduce a new field λ subject to the constraint (linearized)

$$\not{\partial} \lambda + \frac{1}{9} \Gamma^{\mu\nu} \partial_\mu \psi_\nu = 0. \quad (3.16)$$

Hence λ does not represent a new degree of freedom. According to (3.15) it transforms under Q and S supersymmetry:

$$\delta \lambda = -\frac{1}{18} \not{\square} R(e) \epsilon - \frac{1}{3} \Gamma^{\lambda_1 \dots \lambda_7} \epsilon \partial_{\lambda_1} A_{\lambda_2 \dots \lambda_7} + \eta. \quad (3.17)$$

This shows that the introduction of λ does not suffice to remove the non-local terms. We also need a scalar field (linearized)

$$\sigma = -\frac{1}{9} \not{\square} R(e), \quad (3.18)$$

where $R(e)$ is the standard Ricci scalar whose linearized form is

$$R(e) = \not{\square} h_{\mu\mu} - \partial_\mu \partial_\nu h_{\mu\nu}.$$

The variation of (3.18) gives

$$\begin{aligned} \delta \sigma &= -\frac{1}{18} \bar{\epsilon} \not{\square} \Gamma^{\mu\nu} \partial_\mu \psi_\nu + \Lambda_D \\ &= \frac{1}{2} \bar{\epsilon} \lambda + \Lambda_D, \end{aligned} \quad (3.19)$$

so that the non-locality has indeed been absorbed in the new fields λ and σ . Summarizing, the result is

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta \psi_\mu &= \mathcal{D}_\mu(\omega(e)) \epsilon + \Gamma_\mu{}^{\lambda_1 \dots \lambda_7} \epsilon \partial_{\lambda_1} A_{\lambda_2 \dots \lambda_7} - \Gamma_\mu \eta, \end{aligned}$$

$$\begin{aligned}
\delta A_{\lambda_1 \dots \lambda_6} &= \frac{3}{4 \cdot 6!} \{ \bar{\epsilon} \Gamma_{[\lambda_1 \dots \lambda_5} \psi_{\lambda_6]} + \bar{\epsilon} \Gamma_{\lambda_1 \dots \lambda_6} \lambda \} + \partial_{[\lambda_1} \xi_{\lambda_2 \dots \lambda_6]} , \\
\delta \lambda &= \frac{1}{2} (\not{\partial} \sigma) \epsilon - \frac{1}{3} \Gamma^{\lambda_1 \dots \lambda_7} \epsilon \partial_{\lambda_1} A_{\lambda_2 \dots \lambda_7} + \eta , \\
\delta \sigma &= \frac{1}{2} \bar{\epsilon} \lambda .
\end{aligned} \tag{3.20}$$

The algebra closes on all fields with the constraints (linearized)

$$\not{\partial} \lambda + \frac{1}{9} \Gamma^{\mu\nu} \partial_\mu \psi_\nu = 0 , \tag{3.21a}$$

$$\square \sigma + \frac{1}{9} \mathcal{R}(e) = 0 . \tag{3.21b}$$

It is now clear that the linearized theory can be viewed in two different ways. Either we solve (3.21) and substitute the result for λ and σ into the transformation rules. This gives rise to (3.13). Or we express the theory in terms of fields that are inert under S and D transformations. In the linearized theory that is done by making the redefinitions

$$\begin{aligned}
(\psi_\mu)^{\text{new}} &= \psi_\mu + \Gamma_\mu \lambda , \\
(e_\mu^a)^{\text{new}} &= \exp(\sigma) e_\mu^a ,
\end{aligned} \tag{3.22}$$

in which case the restrictions (3.21) can be written as differential constraints expressed entirely in terms of the new fields which coincide with (3.11). This thus leads to the second formulation discussed at the beginning of this section.

In the construction of the complete non-linear theory it is convenient to first define the gauge fields associated with all the local symmetries of the multiplet (3.20). For the dilatations and Lorentz transformations this is rather straightforward, since we can directly reconcile the conformal gravity theory of the previous section with the structure of the transformation rules (3.20) and the constraint (3.21b). The gauge field of the local Lorentz transformations has its usual form in terms of the zehnbein derivative, and the dilatational gauge field b_μ is introduced at the expense of introducing the local conformal boosts. The constraint on σ then takes the form of a conformal d'Alembertian; indeed if we introduce the gauge field f_μ^a associated with K transformations as in the previous section we see that (3.21b) can be rewritten as

$$\square \sigma + f_\mu{}^\mu = 0 , \tag{3.23}$$

which is just the linearized form of the restriction for a would-be compensator field [cf. (2.12) and (2.14)]. We must now also define a gauge field ϕ_μ associated with S supersymmetry. Guided by the experience with four-dimensional superconformal gravity, we choose

$$\phi_\mu = \frac{1}{16} (\Gamma^{\nu\rho} \Gamma_\mu - \frac{7}{9} \Gamma_\mu \Gamma^{\nu\rho}) \partial_\nu \psi_\rho , \tag{3.24}$$

which transforms under S supersymmetry according to

$$\delta_S \phi_\mu = \partial_\mu \eta . \tag{3.25}$$

With this condition the constraint (3.21a) can now be written as

$$\delta\lambda - \Gamma^\mu \phi_\mu = 0, \quad (3.26)$$

so that (3.21a) can be viewed as the linearized form of an appropriately covariantized Dirac equation on λ .

We now assign Weyl weights to all the fields in the multiplet, and make the transformation rules (3.20) consistent with dilatations by introducing appropriate powers of the scalar field $\phi = \exp(w\sigma)$. Choosing the standard weight -1 for e_μ^a all other weights are essentially determined, and we have

$$\begin{aligned} w(e_\mu^a) &= -1, & w(\psi_\mu) &= -\frac{1}{2}, & w(A_{\lambda_1 \dots \lambda_6}) &= 0, \\ w(\lambda) &= +\frac{1}{2}, & w(\phi) &= w. \end{aligned} \quad (3.27)$$

The Q and S transformations of b_μ can be determined by requiring that the commutator of a Q and an S transformation closes in the same manner as on the other fields. For instance on the zehnbein one finds

$$[\delta_Q(\varepsilon), \delta_S(\eta)] = \delta_D(-\frac{1}{2}\bar{\eta}\varepsilon) + \delta_M(\frac{1}{2}\bar{\varepsilon}\Gamma^{ab}\eta). \quad (3.28)$$

This is obtained on b_μ as well if we assign to b_μ the following transformations:

$$\delta b_\mu = \frac{1}{2}\bar{\varepsilon}\phi_\mu - \frac{1}{2}\bar{\eta}\psi_\mu + \partial_\mu A_D + e_\mu^a A_{Ka}, \quad (3.29)$$

where we have included the transformations under dilatations and conformal boosts. Note that the transformations on b_μ are only determined modulo a field-dependent K transformation, because b_μ acts as the compensating field for this symmetry. In four dimensions the transformation of b_μ takes on the same form. There (3.29) can be related to the super-algebra $SU(2, 2|1)$. For the set of bosonic and fermionic symmetries which are operative in the multiplet (3.20) we do not have an underlying gauge algebra. Generalizations of the four-dimensional $SU(2, 2|1)$ algebra have been found for all dimensions allowing Majorana spinors [12], but they contain many more bosonic symmetries than appear to be present in (3.20), and therefore cannot be immediately implemented. It is at present unclear whether or not there exists a relation between the algebras of [12] and our results.

As a first step towards the complete non-linear theory one covariantizes all derivatives in (3.20) with respect to dilatations, local Lorentz transformations, and Q and S supersymmetry. We also introduce covariant curvatures

$$\begin{aligned} R_{\mu\nu}^a(P) &= \mathcal{D}_{[\mu} e_{\nu]}^a - \frac{1}{4}\bar{\psi}_\mu \Gamma^a \psi_\nu, \\ R_{\mu\nu}(Q) &= \mathcal{D}_{[\mu} \psi_{\nu]} - \Gamma_{[\mu}^{\lambda_1 \dots \lambda_7} \psi_{\nu]} R(A)_{\lambda_1 \dots \lambda_7} \phi^{-6/w}, \\ R(A)_{\lambda_1 \dots \lambda_7} &= \partial_{[\lambda_1} A_{\lambda_2 \dots \lambda_7]} - \frac{3}{4 \cdot 6!} \phi^{6/w} \{ \frac{1}{2} \bar{\psi}_{[\lambda_1} \Gamma_{\lambda_2 \dots \lambda_6} \psi_{\lambda_7]} + \bar{\psi}_{[\lambda_1} \Gamma_{\lambda_2 \dots \lambda_7]} \lambda \}, \end{aligned} \quad (3.30)$$

where derivatives \mathcal{D}_μ are covariant with respect to dilatations and Lorentz transformations. One realizes that (3.23) is just the linearized form of the curvature

constraint

$$\Gamma^\mu R_{\mu\nu}(Q) = 0, \quad (3.31)$$

and we now define the full ϕ_μ to be the solution of (3.31). Similarly, ω_μ^{ab} will be the solution of

$$R_{\mu\nu}{}^a(P) = 0, \quad (3.32)$$

with $R(P)$ given by (3.30). The transformation rules of ω_μ^{ab} are now determined by (3.32), and lead to the covariant curvature $R_{\mu\nu}{}^{ab}(M)$. The K gauge field f_μ^a is then determined by

$$R_{\mu\nu}{}^{ab}(M)e^\nu{}_b = 0. \quad (3.33)$$

These three constraints are all of the conventional type because they algebraically restrict certain gauge fields. They are the direct generalizations of the conventional constraints in four-dimensional conformal supergravity.

Before applying the iterative procedure which will lead to the complete transformation rules, we modify (3.20) by adding an $R(A)$ -dependent S supersymmetry transformation. This transformation is chosen in such a way that the final result shows most clearly the relation to Poincaré supergravity with a 6-index gauge field [14]. We shall come back to this relationship in sect. 5.

The complete non-linear transformation rules under Q and S supersymmetry take on the following form:

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu, \\ \delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \frac{1}{4}\phi^{-6/w}(\Gamma_\mu\Gamma^{(7)} - 2\Gamma^{(7)}\Gamma_\mu)\epsilon R(A)_{(7)} \\ &\quad - \frac{21}{16}\bar{\epsilon}\Gamma^a\psi_\mu\Gamma_a\lambda + \frac{1}{1280}\bar{\epsilon}\Gamma^{(5)}\psi_\mu\Gamma_{(5)}\lambda \\ &\quad + \frac{3}{64}(\Gamma_\mu\Gamma^{(3)} + 4\Gamma^{(3)}\Gamma_\mu)\epsilon\bar{\lambda}\Gamma_{(3)}\lambda - \Gamma_\mu\eta, \\ \delta A_{\lambda_1\cdots\lambda_6} &= \frac{3}{4\cdot 6!}\phi^{6/w}\bar{\epsilon}\{\Gamma_{[\lambda_1\cdots\lambda_5}\psi_{\lambda_6]} + \Gamma_{\lambda_1\cdots\lambda_6}\lambda\}, \\ \delta\lambda &= \frac{1}{2w}\phi^{-1}\mathcal{D}\phi\epsilon - \frac{1}{12}\Gamma^{(7)}\epsilon R(A)_{(7)}\phi^{-6/w} + \eta, \\ \delta\phi &= \frac{1}{2}w\bar{\epsilon}\lambda\phi. \end{aligned} \quad (3.34)$$

The derivatives D_μ are completely supercovariant. The conventional constraints retain the same form which implies that the dependent fields ϕ_μ and f_μ^a are changed accordingly by higher-order terms. The constraints (3.21) do develop non-linear modifications other than covariantizations. The fermionic constraint, which is needed to close the algebra on ψ_μ , now reads

$$\mathcal{D}\lambda + \frac{4}{w}\phi^{-1}\mathcal{D}\phi\lambda - \frac{1}{3}\Gamma^{(7)}\lambda R(A)_{(7)}\phi^{-6/w} = 0. \quad (3.35)$$

The bosonic constraint is not needed to close the algebra on the fundamental fields (3.34), but follows from (3.35) by a Q transformation

$$e^\mu{}_a D_\mu (\phi^{-1} D_a \phi) + \frac{4}{w} (\phi^{-1} D_a \phi)^2 + 2240 w \phi^{-12/w} R(A) \cdot R(A) + \frac{4}{3} w \phi^{-6/w} R(A)_{(7)} \bar{\lambda} \Gamma^{(7)} \lambda = 0. \quad (3.36)$$

A further Q variation of (3.36) yields a derivative on (3.35). Of course, (3.35) and (3.36) are S and K invariant. The latter is obvious from the fact that the derivative terms can be written as $\phi^{-4/w} \not{D}(\phi^{4/w} \lambda)$ and $\frac{1}{4} w \phi^{-4/w} (\square \phi^{4/w})$; as discussed in the previous section the Dirac and the d'Alembertian operators are K inert when acting on spinors of weight $\frac{9}{2}$ and scalars of weight 4, respectively.

The algebra of Q and S transformations in (3.34) closes modulo field-dependent transformations and the constraint (3.35). The algebra takes on the following form:

$$\begin{aligned} [\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \frac{1}{2} \bar{\varepsilon}_2 \Gamma^\mu \varepsilon_1 D_\mu + \delta_Q \left(-\frac{21}{16} \bar{\varepsilon}_2 \Gamma^a \varepsilon_1 \Gamma_a \lambda + \frac{1}{1280} \bar{\varepsilon}_2 \Gamma^{(5)} \varepsilon_1 \Gamma_{(5)} \lambda \right) \\ &+ \delta_M \left(\frac{1}{8} \bar{\varepsilon}_2 (\Gamma^{ab} \Gamma^{(7)} + \Gamma^{(7)} \Gamma^{ab} - 4 \Gamma^{[a} \Gamma^{(7)} \Gamma^{b]}) \varepsilon_1 \phi^{-6/w} R(A)_{(7)} \right. \\ &+ \frac{3}{128} \bar{\varepsilon}_2 (\Gamma^{ab} \Gamma^{(3)} + \Gamma^{(3)} \Gamma^{ab} + 8 \Gamma^{[a} \Gamma^{(3)} \Gamma^{b]}) \varepsilon_1 \bar{\lambda} \Gamma_{(3)} \lambda \Big) \\ &+ \delta_A \left(-\frac{3}{4 \cdot 6!} \phi^{6/w} \bar{\varepsilon}_2 \Gamma_{\mu_1 \dots \mu_5} \varepsilon_1 \right) + \delta_S \left\{ \frac{3}{16} \bar{\varepsilon}_2 \Gamma^a \varepsilon_1 \Gamma_a \not{D} \lambda \right. \\ &+ \frac{7}{64} \bar{\varepsilon}_2 \Gamma^a \varepsilon_1 \left(\frac{18}{w} \phi^{-1} \not{D} \phi \Gamma_a \lambda + 3 \Gamma^{(7)} \Gamma_a \lambda \phi^{-6/w} R(A)_{(7)} \right. \\ &+ \frac{24}{w} \Gamma_a (\phi^{-1} \not{D} \phi \lambda - \frac{1}{12} \Gamma^{(7)} \lambda R(A)_{(7)} \phi^{-6/w}) \Big) \\ &\left. - \frac{1}{10240} \bar{\varepsilon}_2 \Gamma^{(5)} \varepsilon_1 \Gamma^a \Gamma_{(5)} \left(\frac{6}{w} \phi^{-1} \not{D} \phi \Gamma_a \lambda + \Gamma^{(7)} \Gamma_a \lambda \phi^{-6/w} R(A)_{(7)} \right) \right\} + \delta_K (\Lambda_1^a), \\ [\delta_Q(\varepsilon), \delta_S(\eta)] &= \delta_D (-\frac{1}{2} \bar{\eta} \varepsilon) + \delta_M (\frac{1}{2} \bar{\varepsilon} \Gamma^{ab} \eta) \\ &+ \delta_S (\frac{1}{32} (-102 (\bar{\eta} \varepsilon) \lambda + 9 (\bar{\eta} \Gamma^{(2)} \varepsilon) \Gamma_{(2)} \lambda + \frac{1}{4} (\bar{\eta} \Gamma^{(4)} \varepsilon) \Gamma_{(4)} \lambda)) + \delta_K (\Lambda_2^a), \\ [\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_K (\bar{\eta}_2 \Gamma^a \eta_1). \end{aligned} \quad (3.37)$$

The commutators of a fermionic and a bosonic transformation, and of two bosonic transformations are not modified. The field-dependent K transformations in (3.37) can be seen only on b_μ and on derived quantities such as curvatures through their dependence on b_μ . Therefore the explicit form of Λ_1^a and Λ_2^a is not of particular interest. For completeness we present some of the transformations of the dependent fields. To determine these one needs the explicit expressions for the covariant

curvatures. The curvatures $R(Q)$ and $R(M)$ take on the form

$$\begin{aligned} R_{\mu\nu}(Q) = & \mathcal{D}_{[\mu}\psi_{\nu]} - \Gamma_{[\mu}\phi_{\nu]} + \frac{1}{4}(\Gamma_{[\mu}\Gamma^{(7)} - 2\Gamma^{(7)}\Gamma_{\mu]})\psi_{\nu]}\phi^{-6/w}R(A)_{(7)} \\ & + \frac{21}{32}\bar{\psi}_{[\mu}\Gamma^a\psi_{\nu]}\Gamma_a\lambda - \frac{1}{2560}\bar{\psi}_{[\mu}\Gamma^{(5)}\psi_{\nu]}\Gamma_{(5)}\lambda \\ & + \frac{3}{64}(\Gamma_{[\mu}\Gamma^{(3)} + 4\Gamma^{(3)}\Gamma_{\mu]})\psi_{\nu]}\bar{\lambda}\Gamma_{(3)}\lambda, \end{aligned} \quad (3.38)$$

$$\begin{aligned} R_{\mu\nu}{}^{ab}(M) = & \partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 2e_{[\mu}{}^{[a}f_{\nu]}{}^{b]} + \frac{1}{2}\bar{\psi}_{[\mu}\Gamma^{ab}\phi_{\nu]} \\ & - \bar{\psi}_{[\mu}\{\frac{1}{2}\Gamma_{\nu]}\mathbf{R}_{ab}(Q) + \Gamma^{[a}\mathbf{R}_{\nu]}{}^{b]}(Q) \\ & + \frac{1}{8}(\Gamma^{ab}\Gamma^{(7)} + \Gamma^{(7)}\Gamma^{ab} - 4\Gamma^{[a}\Gamma^{(7)}\Gamma^{b]})\psi_{\nu]}\phi^{-6/w}R(A)_{(7)} \\ & + \frac{3}{128}(\Gamma^{ab}\Gamma^{(3)} + \Gamma^{(3)}\Gamma^{ab} + 8\Gamma^{[a}\Gamma^{(3)}\Gamma^{b]})\psi_{\nu]}\bar{\lambda}\Gamma_{(3)}\lambda, \end{aligned} \quad (3.39)$$

while $R(P)$ is given by (3.30). The transformation rules of $\omega_\mu{}^{ab}$ are (Q , S and K)

$$\begin{aligned} \delta\omega_\mu{}^{ab} = & -\frac{1}{2}\bar{\epsilon}\Gamma^{ab}\phi_\mu + \frac{1}{2}\bar{\epsilon}\Gamma_\mu\mathbf{R}_{ab}(Q) + \bar{\epsilon}\Gamma^{[a}\mathbf{R}_\mu{}^{b]}(Q) \\ & + \frac{1}{8}\bar{\epsilon}(\Gamma^{ab}\Gamma^{(7)} + \Gamma^{(7)}\Gamma^{ab} - 4\Gamma^{[a}\Gamma^{(7)}\Gamma^{b]})\psi_\mu\phi^{-6/w}R(A)_{(7)} \\ & + \frac{3}{128}\bar{\epsilon}(\Gamma^{ab}\Gamma^{(3)} + \Gamma^{(3)}\Gamma^{ab} + 8\Gamma^{[a}\Gamma^{(3)}\Gamma^{b]})\psi_\mu\bar{\lambda}\Gamma_{(3)}\lambda \\ & - \frac{1}{2}\bar{\eta}\Gamma^{ab}\psi_\mu + 2\Lambda_K{}^{[a}e_\mu{}^{b]}. \end{aligned} \quad (3.40)$$

The explicit form for ϕ_μ in terms of the fundamental fields is easily obtained from (3.31) using the linearized form (3.24) and the complete expression (3.38) for $R_{\mu\nu}(Q)$. The S and K transformations of ϕ_μ are

$$\begin{aligned} \delta\phi_\mu = & \mathcal{D}_\mu\eta - \frac{1}{12}(2\Gamma_\mu\Gamma^{(7)} - 3\Gamma^{(7)}\Gamma_\mu)\eta\phi^{-6/w}R(A)_{(7)} \\ & + \frac{1}{64}(4\Gamma_\mu\Gamma^{(3)} + 9\Gamma^{(3)}\Gamma_\mu)\eta\bar{\lambda}\Gamma_{(3)}\lambda \\ & + \psi_\mu \text{ dependent terms} + \frac{1}{2}\Lambda_K{}^a\Gamma_a\psi_\mu. \end{aligned} \quad (3.41)$$

We will not give the Q transformation rule, which is quite lengthy, nor the transformation rules of $f_\mu{}^a$.

In the derivation of (3.34) and (3.37) one needs the variations of $R(A)$ and $\phi^{-1}D_a\phi$. For the convenience of the reader we give these explicitly (Q , S and K transformations):

$$\begin{aligned} \delta(\phi^{-1}D_a\phi) = & \frac{1}{2}w\bar{\epsilon}D_a\lambda - \frac{1}{8}w\phi^{-6/w}\bar{\epsilon}(2\Gamma^a\Gamma^{(7)} - \Gamma^{(7)}\Gamma^a)\lambda R(A)_{(7)} \\ & - \frac{1}{2}w\bar{\eta}\Gamma_a\lambda - w\Lambda_{Ka}, \end{aligned} \quad (3.42)$$

$$\begin{aligned} \delta R(A)_{a_1\cdots a_7} = & \frac{3}{4 \cdot 6!}\phi^{6/w}[-\bar{\epsilon}\Gamma_{[a_1\cdots a_5}\mathbf{R}_{a_6a_7]}(Q) + \bar{\epsilon}\Gamma_{[a_1\cdots a_6}\phi^{-6/w}D_{a_7]}(\phi^{6/w}\lambda) \\ & + \frac{3}{64}\bar{\lambda}\Gamma_{[a_1\cdots a_6}(\Gamma_{a_7]}\Gamma^{(3)} + 4\Gamma^{(3)}\Gamma_{a_7])\epsilon\bar{\lambda}\Gamma_{(3)}\lambda \\ & + \frac{1}{4}\bar{\lambda}\Gamma_{[a_1\cdots a_6}(\Gamma_{a_7]}\Gamma^{(7)} - 2\Gamma^{(7)}\Gamma_{a_7])\epsilon\phi^{-6/w}R(A)_{(7)} - \bar{\eta}\Gamma_{a_1\cdots a_7}\lambda]. \end{aligned} \quad (3.43)$$

We have presented the complete off-shell supergravity theory in terms of a multiplet of fields, involving two unconventional constraints. We remark that the constraint (3.36) on ϕ is just the supersymmetric generalization of the constraint (2.14), which we have introduced in conformal gravity to be able to construct invariant actions in arbitrary dimensions. In the superconformal context the would-be compensating fields ϕ and λ are already required to obtain the irreducible, massive field representation of the supersymmetry algebra containing the gravitational field.

As in conformal gravity, the would-be compensators also enable us to write down invariant actions. To conclude this section we construct an invariant for the fields of the linearized superconformal multiplet. We start from a lagrangian which is invariant under the linearized transformation rules (3.13). It takes on the form

$$\begin{aligned} \mathcal{L} = & -\frac{7}{32}(R_{\mu\nu}(e)R_{\mu\nu}(e) - \frac{5}{18}R^2(e)) \\ & -\frac{7}{64}(\bar{\psi}_{\mu\nu}\not{\partial}\psi_{\mu\nu} - \frac{1}{18}\bar{\psi}_{\mu\nu}\Gamma^{\mu\nu}\not{\partial}\Gamma^{\lambda\rho}\psi_{\lambda\rho}) \\ & + 2205R(A)_{\lambda_1\cdots\lambda_7}\square R(A)^{\lambda_1\cdots\lambda_7}. \end{aligned} \quad (3.44)$$

Here $\psi_{\mu\nu} = \partial_{[\mu}\psi_{\nu]}$. The gravitational part corresponds to (2.19) for $d = 10$. To proceed from (3.44) to an action that is invariant under the transformation rules (3.34) is in principle straightforward, but algebraically complicated. One first brings (3.44) in a form which contains the full superconformal curvatures, and which is invariant under dilatations and conformal boosts. The result is

$$\begin{aligned} e^{-1}\mathcal{L} = & \frac{3}{32}e^\mu_a e^\nu_b e^\lambda_c e^\rho_d R_{\mu\nu}{}^{ab}(M)R_{\lambda\rho}{}^{cd}(M)\phi^{6/w} - \bar{R}^{\mu\nu}(Q)R_{\mu\nu}(S)\phi^{6/w} + 2205R(A)_{a_1\cdots a_7} \\ & \times \left(\square R(A)_{a_1\cdots a_7} - \frac{16}{w}(\phi^{-1}D_{[a_1}\phi)D^{a_8}R(A)_{a_2\cdots a_8]} \right) \phi^{-6/w}. \end{aligned} \quad (3.45)$$

However, (3.45) is not invariant under the full Q and S transformations (3.34), because the non-linear terms in (3.34) generate non-linear contributions to the transformation rules of the curvatures, which require additional terms in (3.45). We shall not complete this analysis, but merely point out that there is no essential obstacle in the extension of (3.45) to a full superconformal invariant. Thus the situation is similar to $d = 4$, $N = 3, 4$ where only the terms quadratic in the fields in the superconformal action are known.

4. Dimensional reduction of ten-dimensional conformal supergravity

The main difference between ten-dimensional conformal supergravity and its four-dimensional counterpart is that the first theory has differential constraints. However, in both cases conformal supergravity is based on a unique irreducible supermultiplet involving graviton and gravitino states, and because these multiplets have an equal number of states we expect that they are related by means of a rather straightforward dimensional reduction. This section is therefore devoted to reductions of $d = 10$ conformal supergravity in order to exhibit how the differential

constraints are resolved. Since superconformal gravity has not yet been formulated in five dimensions, we prefer to concentrate on the reduction to five dimensions. The theory that emerges has unconstrained fields, and turns out to be gauge equivalent to the five-dimensional theory of Howe and Lindström [10]. This is to be expected because the latter is also based on an irreducible multiplet of $128 + 128$ degrees of freedom, and it was in fact obtained from the five-dimensional supercurrent. We should add that the corresponding problem of the reduction of the ten-dimensional supercurrent has been discussed in [8].

In the reduction to five dimensions the fields only depend on the coordinates of a five-dimensional Minkowski subspace. Vector indices in this space are denoted by μ ($\mu = 1, \dots, 5$) and the remaining ones by s ($s = 1, \dots, 5$). Consequently we restrict the ten-dimensional Lorentz transformations to $SO(4, 1) \times SO(5)$, which act on the spinors according to the covering group $Sp(2, 2) \times Sp(4)$. The decomposition of the gamma matrices into two mutually commuting sets can be expressed by the tensor products

$$\begin{aligned} \Gamma^\mu &= \sigma_1 \otimes \gamma^\mu \otimes \mathbb{1}, & (\mu = 1, \dots, 5), \\ \Gamma^{5+s} &= \sigma_2 \otimes \mathbb{1} \otimes \gamma^s, & (s = 1, \dots, 5), \end{aligned} \quad (4.1)$$

where σ_1 and σ_2 are the Pauli matrices, and γ_μ and γ_s are the gamma matrices in five dimensions. Choosing a representation in which

$$\begin{aligned} \gamma_{[s} \gamma_t \gamma_u \gamma_v \gamma_w] &= \mathbb{1} \varepsilon_{stuvw}, \\ \gamma_{[\mu} \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau] &= \mathbb{1} \varepsilon_{\mu\nu\rho\sigma\tau}, \end{aligned} \quad (4.2)$$

we have

$$\Gamma^{11} = -\sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}. \quad (4.3)$$

Since the 32-component spinors of the ten-dimensional theory are chiral, (4.3) shows that they take values in either the first or the last sixteen components. Hence we can use 16-dimensional spinors, in which case the σ -matrices no longer occur explicitly. The spinors remain Majorana fields, and the 16×16 charge conjugation matrix C is symmetric and satisfies $C^{-1} \gamma_\mu C = \gamma_\mu^T$, $C^{-1} \gamma_s C = \gamma_s^T$.

With these definitions it is straightforward to obtain the linearized transformation rules in five dimensions. The definition of the fields with their $Sp(4)$ or $SO(5)$ representation content and the linearized gauge transformations is indicated in table 1. The linearized supersymmetry transformations follow from (3.34) and read as follows:

$$\begin{aligned} \delta h_{\mu\nu} &= \frac{1}{4} \bar{\varepsilon} (\gamma_\mu \psi_\nu + \gamma_\nu \psi_\mu), \\ \delta h_\mu{}^s &= \frac{1}{4} \bar{\varepsilon} (\gamma_\mu \psi^s - i \gamma^s \psi_\mu), \\ \delta h^{st} &= -\frac{1}{4} i \bar{\varepsilon} (\gamma^s \psi^t + \gamma^t \psi^s), \end{aligned}$$

TABLE 1
Decomposition of the $d = 10$ conformal supergravity fields to five dimensions

$d = 10$	$d = 5$	SO(5) or Sp(4)	Gauge transformations
$h_{\hat{\mu}\hat{\nu}}$	$h_{\mu\nu}$ h_μ^s h^{st}	1 5 14 + 1	$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \Lambda_D \delta_{\mu\nu}$ $\delta h_\mu^s = \partial_\mu \Lambda^s$ $\delta h^{st} = -\delta^{st} \Lambda_D$
σ	σ	1	$\delta \sigma = \Lambda_D$
$A_{\hat{\mu}_1 \dots \hat{\mu}_6}$	A_μ $A_{\mu\nu}^s$ $A_{\mu\nu}^{st}$ A_μ^{st}	1 5 10 10	$\delta A_\mu = \partial_\mu \xi$ $\delta A_{\mu\nu}^s = \partial_{[\mu} \xi_{\nu]}^s$ $\delta A_{\mu\nu}^{st} = \partial_\lambda \xi_{[\mu\nu\lambda]}^{st}$ $\delta A_\mu^{st} = \partial_\lambda \xi_{[\mu\lambda]}^{st}$
$\psi_{\hat{\mu}}$	ψ_μ ψ^s	4 16 + 4	$\delta \psi_\mu = \partial_\mu \varepsilon - \gamma_\mu \eta$ $\delta \psi^s = -i \gamma^s \eta$
λ	λ	4	$\delta \lambda = \eta$

$$\delta \sigma = \frac{1}{2} \bar{\varepsilon} \lambda,$$

$$\delta A_\mu = \frac{1}{8} \bar{\varepsilon} (\gamma_\mu \gamma_s \psi^s + i \psi_\mu + 6 i \gamma_\mu \lambda),$$

$$\delta A_{\mu\nu}^s = \frac{1}{16} \bar{\varepsilon} (-i \gamma_{\mu\nu} \gamma^{st} \psi_t + 2 \gamma^s \gamma_{[\mu} \psi_{\nu]} + 6 \gamma^s \gamma_{\mu\nu} \lambda),$$

$$\delta A_{\mu\nu}^{st} = \frac{1}{16} \bar{\varepsilon} (\gamma_{\mu\nu} \gamma^{stu} \psi_u + i \gamma_{\mu\nu\rho} \gamma^{st} \psi_\rho + 6 i \gamma_{\mu\nu} \gamma^{st} \lambda),$$

$$\delta A_\mu^{st} = \frac{1}{16} \bar{\varepsilon} (-2 i \gamma_\mu \gamma^{[s} \psi^{t]} + \gamma_{\mu\nu} \gamma^{st} \psi_\nu + 6 \gamma_\mu \gamma^{st} \lambda),$$

$$\delta \psi_\mu = \mathcal{D}_\mu \varepsilon - \frac{1}{2} i \not{\partial} h_\mu^s \gamma_s \varepsilon + \frac{1}{4} (\gamma_\mu R^- - 2 R^+ \gamma_\mu) \varepsilon,$$

$$\delta \psi^s = -\frac{1}{2} \gamma^{\mu\nu} \partial_{[\mu} h_{\nu]}^s \varepsilon + \frac{1}{2} i \not{\partial} h^{st} \gamma_t \varepsilon + \frac{1}{4} i (\gamma^s R^- + 2 R^+ \gamma^s) \varepsilon,$$

$$\delta \lambda = \frac{1}{2} \not{\partial} \sigma \varepsilon - \frac{1}{12} R^- \varepsilon, \quad (4.4)$$

where we have used the definition

$$R^\pm = \pm i \gamma^{\mu\nu} \partial_{[\mu} A_{\nu]} - \frac{1}{2} \varepsilon^{\mu_1 \dots \mu_3 \rho \sigma} \gamma_{\rho \sigma} \gamma_s \partial_{\mu_1} A_{\mu_2 \mu_3}^s \\ \pm i \gamma^\nu \gamma_{st} \partial_\lambda A_{\lambda\nu}^{st} + \gamma_{st} \partial \cdot A^{st}. \quad (4.5)$$

As we shall see shortly $V_\mu^{st} \equiv \partial_\mu A_{\nu\mu}^{st}$ will play the role of the gauge field associated with the Sp(4) internal symmetry. Therefore, we redefine ψ_μ such that V_μ^{st} occurs in its Q variation as part of an Sp(4) covariant derivative on the parameter ε . This leads to a corresponding redefinition of $h_{\mu\nu}$ in order that it transforms under supersymmetry into the redefined Rarita-Schwinger field only. Both redefinitions are unique if one requires that the S and D transformations remain unaffected, and we find

$$\psi_\mu \rightarrow (\psi_\mu)^{\text{new}} = \psi_\mu - \frac{3}{4} i \gamma_\mu (\gamma_s \psi^s + 5 i \lambda), \\ h_{\mu\nu} \rightarrow (h_{\mu\nu})^{\text{new}} = h_{\mu\nu} + \frac{3}{4} \delta_{\mu\nu} (h^{ss} + 5 \sigma). \quad (4.6)$$

In terms of these fields the differential constraints (3.35), (3.36) take the form

$$\begin{aligned} \gamma^{\mu\nu} \partial_\mu \psi_\nu + 2i \not{\partial} (\gamma_s \psi^s + 3i\lambda) &= 0, \\ \square h_{\mu\mu} - \partial_\mu \partial_\nu h_{\mu\nu} - 2\square(h^{ss} + 3\sigma) &= 0. \end{aligned} \quad (4.7)$$

We identify the lowest-dimensional spinor Λ by the requirement that it does not transform into V_μ^{st} . This yields

$$\Lambda = \gamma_s \psi^s + 9i\lambda. \quad (4.8)$$

The lowest-dimensional real scalar whose supersymmetry variation leads to Λ is given by

$$C = h_{ss} + 9\sigma. \quad (4.9)$$

The identification of the remaining components of the supermultiplet follows straightforwardly from considering successive supersymmetry variations starting with $\delta\Lambda$. It is then crucial to verify that linear combinations of the restricted fields independent of (4.8)–(4.9) occur only with appropriate powers of $\not{\partial}$, so that they can be replaced by derivatives on vierbein and gravitino fields without the need for introducing inverse powers of \square or $\not{\partial}$.

We briefly describe the result of this procedure. The supersymmetry transformation of Λ yields a scalar E^{st} and a tensor $T_{\mu\nu}^s$ defined by

$$\begin{aligned} E^{st} &= \partial \cdot A^{st}, \\ T_{\mu\nu}^s &= \partial_{[\mu} h_{\nu]}^s - i\varepsilon_{\mu\nu}{}^{\rho\sigma\tau} \partial_\rho A_{\sigma\tau}^s. \end{aligned} \quad (4.10)$$

Note that $T_{\mu\nu}^s$ is not subject to a Bianchi identity since it consists of a linear combination of a curvature and a dual curvature. The fact that E and T are invariant under the remnants of the gauge transformations of the 6-index gauge field $A_{\mu_1 \dots \mu_6}$ (see table 1) is crucial in order to avoid minimal coupling inconsistencies which arise when E and T are forced to transform under *local* $SO(5)$ [or $Sp(4)$].

The transformations of E and T do indeed lead to the constrained spinor field, but in such a way that the corresponding terms can be eliminated by using (4.7) without introducing inverse powers of $\not{\partial}$. Both E and T transform into a new fermionic component

$$\chi^s = \not{\partial} (\psi^s - \tfrac{1}{5} \gamma^s \gamma_\mu \psi^\mu), \quad (4.11)$$

whose variation leads straightforwardly to two new quantities, namely

$$\begin{aligned} D^{st} &= \square(h^{st} - \tfrac{1}{5} \delta^{st} h^{\mu\mu}), \\ R_{\mu\nu}^{st}(V) &= \partial_{[\mu} V_{\nu]}^{st}. \end{aligned} \quad (4.12)$$

The variation of D^{st} leads directly to $\not{\partial} \chi^s$, so what remains is to verify whether the supersymmetry variations of V_μ^{st} and ψ_μ are expressible in terms of the previous

components, thereby avoiding undesirable terms depending on constrained fields, or on fields involved in the definitions (4.10)–(4.12) but without the corresponding derivative structure. This is indeed the case for $\delta\psi_\mu$ except for terms proportional to ∂_μ that can be interpreted as field-dependent Q supersymmetry transformations. Also the variation of the conserved vector V_μ^{st} cannot directly be cast in the desired form, but the improper terms can all be written with an overall derivative ∂_μ :

$$\delta V_\mu^{st} = \frac{1}{8}\bar{\epsilon}(\gamma_\mu \gamma^{[s} \chi^{t]} - \frac{3}{2}i\gamma^{st}\phi_\mu + \frac{1}{40}\gamma^{st}\gamma_\mu \not{D}\Lambda) + \partial_\mu \text{ terms}, \quad (4.13)$$

where ϕ_μ is the gauge field of S supersymmetry to be defined shortly. This forces us to interpret V_μ^{st} as the gauge field of $\text{Sp}(4)$.

We now give the supersymmetry transformations of the five-dimensional superconformal theory:

$$\begin{aligned} \delta C &= -\frac{1}{2}i\bar{\epsilon}\Lambda, \\ \delta\Lambda &= \frac{1}{2}i\not{D}C\epsilon - \frac{1}{2}\gamma^{\mu\nu}T_{\mu\nu}{}^s\gamma_s\epsilon + iE^{st}\gamma_{st}\epsilon - 2\gamma^{\mu\nu}\partial_\mu A_\nu\epsilon, \\ \delta A_\mu &= \frac{1}{8}\bar{\epsilon}(i\psi_\mu + \frac{1}{4}\gamma_\mu\Lambda), \\ \delta E^{st} &= \frac{1}{16}\bar{\epsilon}(-2i\gamma^{[s}\chi^{t]} + \frac{1}{5}\gamma^{st}\not{D}\Lambda), \\ \delta T_{\mu\nu}{}^s &= -\frac{1}{8}\bar{\epsilon}\gamma_{\mu\nu}\chi^s - \frac{1}{2}i\bar{\epsilon}\gamma^s R_{\mu\nu}(Q) - \frac{1}{16}\bar{\epsilon}\gamma^s\gamma_{[\mu}D_{\nu]}\Lambda + \frac{1}{80}\bar{\epsilon}\gamma^s\gamma_{\mu\nu}\not{D}\Lambda, \\ \delta V_\mu^{st} &= \frac{1}{8}\bar{\epsilon}(\gamma_\mu \gamma^{[s}\chi^{t]} - \frac{3}{2}i\gamma^{st}\phi_\mu + \frac{1}{40}\gamma^{st}\gamma_\mu \not{D}\Lambda), \\ \delta\chi^s &= \frac{1}{2}iD^{st}\gamma_t\epsilon + \frac{1}{2}\gamma^{\rho\sigma}T_{\rho\sigma}{}^s\tilde{\delta}\epsilon + 2\gamma^{\rho\sigma}R_{\rho\sigma}{}^{st}(V)\gamma_t\epsilon \\ &\quad - 2iE^{st}\gamma_t\epsilon - \text{trace}, \\ \delta D^{st} &= -\frac{1}{4}i\bar{\epsilon}(\gamma^s\chi^t + \gamma^t\chi^s), \\ \delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \frac{1}{2}i(2\gamma_\mu\gamma^{\rho\sigma} - \gamma^{\rho\sigma}\gamma_\mu)\partial_\rho A_{\sigma\epsilon} \\ &\quad + \frac{1}{8}i(\gamma_\mu\gamma^{\rho\sigma} + 2\gamma^{\rho\sigma}\gamma_\mu)\gamma_s\epsilon T_{\rho\sigma}{}^s, \\ \delta h_{\mu\nu} &= \frac{1}{4}\bar{\epsilon}(\gamma_\mu\psi_\nu + \gamma_\nu\psi_\mu), \end{aligned} \quad (4.14)$$

where the gauge field of S supersymmetry is given by

$$\phi_\mu = \frac{1}{3}\gamma_\mu{}^{\rho\sigma}\partial_\rho\psi_\sigma - \frac{1}{4}\gamma_\mu\gamma^{\rho\sigma}\partial_\rho\psi_\sigma.$$

The derivative \mathcal{D}_μ is covariant with respect to dilatations, $\text{Sp}(4)$ and Lorentz transformations, whereas D_μ is the full superconformal covariant derivative. The D and S transformations have inhomogeneous terms when acting on some of the fields:

$$\begin{aligned} \delta C &= 4\Lambda_D, \\ \delta h_{\mu\nu} &= -\delta_{\mu\nu}\Lambda_D, \\ \delta\Lambda &= 4i\eta, \\ \delta\psi_\mu &= -\gamma_\mu\eta. \end{aligned} \quad (4.15)$$

Therefore the corresponding covariant curvatures and derivatives acquire extra terms at the linearized level:

$$\begin{aligned} R_{\mu\nu}(Q) &= \partial_{[\mu}\psi_{\nu]} - \gamma_{[\mu}\phi_{\nu]}, \\ D_{\mu}\Lambda &= \partial_{\mu}\Lambda - 4i\phi_{\mu}. \end{aligned} \quad (4.16)$$

A noteworthy feature of the $d = 5$ Weyl multiplet is that it contains central charge transformations $\delta_Z A_{\mu} = \partial_{\mu}z$, which are consistent with the conformal algebra because they are modified by the presence of the scalar field C :

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] \rightarrow \delta_Z(\tfrac{1}{8}i \exp(\tfrac{1}{4}C)\bar{\varepsilon}_2\varepsilon_1). \quad (4.17)$$

One can make C - and Λ -dependent redefinitions of the fields such that all fields except C and Λ become inert under dilatations and S supersymmetry. Or alternatively, one imposes a D and S gauge condition $C = \Lambda = 0$. In this way one finds the $d = 5$ supergravity theory constructed in [10]. Hence this theory and the $d = 5$ superconformal theory are simply gauge equivalent.

In the last crucial step of this reduction a conserved vector was reinterpreted as an unconstrained gauge field. In that way the undesirable terms in the supersymmetry variation involving inverse powers of \not{x} could be dropped because they took the form of a field-dependent gauge transformation. The reader may wonder whether such a trick would not be applicable in the original ten-dimensional theory in order to eliminate the differential constraints (3.35), (3.36). More precisely, would it be possible to reinterpret the dual curvature $\tilde{R}(A)$, subject to the constraint (Bianchi identity)

$$\partial^{\mu}\tilde{R}(A)_{\mu\nu\rho} = 0 \quad (4.18)$$

as a gauge field with gauge transformations

$$\delta\tilde{R}(A)_{\mu\nu\rho} = \partial_{[\mu}\xi_{\nu\rho]}? \quad (4.19)$$

In this case one could interpret the non-local terms in the supersymmetry variation of $R(A)$ as field-dependent transformations of the form (4.19). However, this interpretation leads to immediate difficulties; when we apply (4.19) to the variation $\delta\psi_{\mu}$ the result does not take the form of a supersymmetry transformation as in the five-dimensional case. Hence it seems impossible to introduce the gauge transformation (4.19) without affecting the closure of the gauge algebra. One way to overcome this difficulty might be the introduction of extra gauge fields, in which case one could presumably make contact with the $d = 10$ superconformal algebra presented in [12]. This interesting possibility to avoid the $d = 10$ differential constraints altogether deserves further study.

It is clear how the differential constraints are avoided in the reduction. In lower dimension the fields split up in a large variety of matter fields and gauge fields, which allow one to absorb the differential operators rather easily. The subtle role

played by the field strengths which must be reinterpreted in terms of unconstrained fields makes it difficult to foresee what happens in the reduction to $d > 5$ dimensions.

The reduction to $d = 4$ proceeds in a similar fashion as to $d = 5$. The vector indices of the four-dimensional Minkowski space are denoted by μ ($\mu = 1, \dots, 4$) and the remaining ones by s ($s = 1, \dots, 6$). Consequently we restrict the ten-dimensional Lorentz transformations to $\text{SO}(3, 1) \times \text{SO}(6)$ with $\text{SL}(2, \mathbb{C}) \times \text{SU}(4)$ as the corresponding covering group. The definition of the four-dimensional fields is indicated in table 2, together with their $\text{SO}(6)$ or $\text{SU}(4)$ representation content and linearized gauge transformations.

It is useful to redefine ψ_μ and $h_{\mu\nu}$ in a similar way as in (4.6). The lowest dimensional scalars and spinor of the theory are now easy to identify, because $d = 4$ conformal supergravity is known to have two scalars with zero Weyl weight and an S invariant spinor. Hence one starts from the scalars A and B , where the latter is defined by

$$B = \sigma + \frac{1}{6} h^{ss}, \quad (4.20)$$

which both transform into an S invariant spinor Λ according to

$$\delta(B - \frac{2}{3} i \Lambda) = \frac{1}{2} \bar{\epsilon} (1 + \gamma_5) \Lambda. \quad (4.21)$$

The fact that we have two scalar fields at this point is crucially related to the fact that the $d = 10$ conformal theory uniquely selects a 6-index antisymmetric tensor gauge field. A 2-index field, which corresponds to a field strength dual to $R(A)$, would instead lead to a single 2-index tensor in $d = 4$, which is conjugate to a scalar. It should be clear that the $d = 4$ and $d = 10$ conformal supergravities are based on unique field representations and that the conversion of tensor gauge fields of different rank by duality transformations must take place in the context of field theories based on more degrees of freedom such as Poincaré supergravity.

TABLE 2
Decomposition of the $d = 10$ conformal supergravity fields to four dimensions

$d = 10$	$d = 4$	$\text{SO}(6)$ or $\text{SU}(4)$	Gauge transformations
$h_{\hat{\mu}\hat{\nu}}$	$h_{\mu\nu}$	1	$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \Lambda_D \delta_{\mu\nu}$
	h_μ^s	6	$\delta h_\mu^s = \partial_\mu \Lambda^s$
	h^{st}	$20' + 1$	$\delta h^{st} = -\delta^{st} \Lambda_D$
σ	σ	1	$\delta \sigma = \Lambda_D$
	A	1	$\delta A = 0$
	A_μ^s	6	$\delta A_\mu^s = \partial_\mu \xi^s$
$A_{\hat{\mu}_1 \dots \hat{\mu}_6}$	$A_{\mu\nu}^{st}$	15	$\delta A_{\mu\nu}^{st} = \partial_{[\mu} \xi_{\nu]}^{st}$
	A_μ^{sttu}	20	$\delta A_\mu^{sttu} = \partial_\nu \xi_{[\mu\nu]}^{sttu}$
	$\psi_{\hat{\mu}}$	4	$\delta \psi_\mu = \partial_\mu \epsilon - \gamma_\mu \eta$
λ	ψ^s	$20 + 4$	$\delta \psi^s = -\gamma^s \gamma_5 \eta$
	λ	4	$\delta \lambda = \eta$

We will return to this in the context of Yang–Mills–Einstein supergravity discussed in sect. 5.

From this point on the identification of the remaining components of the $d = 4$ supermultiplet is completely analogous to the $d = 5$ reduction, and we refrain from giving further details. At the end one may rewrite the linearized transformation rules in an $SU(4)$ covariant way. In order to do this one must find an $SU(4)$ covariant representation of the Clifford algebra generated by the gamma matrices γ_s ($s = 1, \dots, 6$). How such a representation can be obtained is explained in [13]. The result yields indeed the $SU(4)$ covariant multiplet corresponding to $N = 4$, $d = 4$ conformal supergravity [4].

5. The relation with Poincaré supergravity

The ten-dimensional superconformal theory is based on the fields e_μ^a , ψ_μ , $A_{\mu_1 \dots \mu_6}$, λ and ϕ . On-shell Poincaré supergravity [14] can be formulated in terms of the same fields. It is also possible to write the Q supersymmetry transformations of the superconformal multiplet in such a way that they take on the same form as the Poincaré supersymmetry transformations, and this was achieved in sect. 3 by including a suitable field-dependent S transformation. This shows that in ten dimensions there must be an intricate relation between (on-shell) Poincaré and (off-shell) conformal supergravity. In the superconformal theory the algebra (3.37) closes on field-dependent superconformal transformations, and on the constraint which restricts λ . As the transformation rules are the same, the calculation of the $[Q, Q]$ commutator in Poincaré supergravity is identical to the superconformal calculation, and therefore both the S parameter in (3.37) and the constraint (3.35) should be related to the non-closure terms in the Poincaré case which are proportional to the Poincaré field equations. In the present section we will discuss various aspects of this relation between Poincaré and conformal supergravity.

Let us begin by considering a counting argument which indicates that this near equality of the two theories is a special property of $d = 10$. In $d = 10$ the superconformal multiplet contains $128 + 128$ components. The super-Poincaré theory is described in terms of fields that consist of 130 bosonic (45, 84 and 1 for e_μ^a , $A_{\mu_1 \dots \mu_6}$ and ϕ , respectively) and 160 fermionic (144 and 16 for ψ_μ and λ) field degrees of freedom. Therefore, the number of field components of the Poincaré theory (counted as off-shell degrees of freedom) exceeds that of the superconformal multiplet. It might therefore be possible to impose restrictions on the Poincaré fields without the necessity of going completely on-shell. The counting argument also indicates what form these restrictions should take. One must eliminate 2 bosonic and 32 fermionic degrees of freedom, which can be done by introducing an additional symmetry and a constraint for both fermions and bosons. This then yields the superconformal multiplet. It is interesting to note that this particular relation between Poincaré and conformal supergravity holds only if Poincaré

supergravity is formulated with a 6-index antisymmetric gauge field. If one uses a 2-index gauge field to describe the 28 on-shell degrees of freedom, the number of bosonic field components is only 82, which is less than 128.

In dimensions other than 10 the previous counting argument yields a different result: the number of components of Poincaré supergravity is always smaller than that of the smallest off-shell multiplet containing spin 2. In $d = 11$ for instance the Poincaré theory is realized in terms of 175 bosonic and 320 fermionic components, while the smallest off-shell multiplet in $d = 11$ (or in $N = 8$, $d = 4$) has 2^{16} components. In extended supergravity in $d = 4$ the Weyl multiplets ($N \leq 4$) have $(5 - N)2^{2N}$ degrees of freedom, while the on-shell Poincaré supergravity multiplets always have fewer components. The $d = 4$ result can be understood by reduction from $d = 10$. The $d = 10$ conformal multiplet gives the $N = 4$, $d = 4$ superconformal theory, which again has $128 + 128$ degrees of freedom. But $d = 10$ Poincaré supergravity gives on reduction the $N = 4$, $d = 4$ supergravity multiplet coupled to matter. The pure supergravity multiplet then has fewer degrees of freedom than the superconformal multiplet.

Poincaré supergravity in $d = 10$ is described by the lagrangian density

$$\begin{aligned}
 e^{-1} \mathcal{L}(N = 1, d = 10) = & -R(e, \omega(e)) - \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\rho\sigma} \mathcal{D}_\rho(\omega(e)) \psi_\sigma \\
 & - 4 \cdot 7! \phi^{-12/w} R^0(A)^{\mu_1 \dots \mu_7} R^0(A)_{\mu_1 \dots \mu_7} - 36 \bar{\lambda} \Gamma^\mu \mathcal{D}_\mu(\omega(e)) \lambda \\
 & - \frac{36}{w^2} (\phi^{-1} \partial_\mu \phi)^2 + \frac{36}{w} \bar{\psi}_\mu (\phi^{-1} \not{\partial} \phi) \Gamma^\mu \lambda \\
 & + \frac{1}{2} \phi^{-6/w} R^0(A)_{(7)} (\bar{\psi}_\mu \Gamma^{[\mu} \Gamma^{(7)} \Gamma^{\nu]} \psi_\nu + 12 \bar{\psi}_\mu \Gamma^{(7)} \Gamma^\mu \lambda) \\
 & + \text{four-fermion terms} ,
 \end{aligned} \tag{5.1}$$

where $R^0(A)$ denotes the field strength of the tensor gauge field $A_{\mu_1 \dots \mu_6}$ *without* supercovariantizations. The lagrangian (5.1) differs from the one in [15, 17] by the use of a 6-index instead of a 2-index antisymmetric tensor [14]. The two versions are related by a duality transformation as we will discuss later. Compared to [15] λ has been redefined with a multiplicative factor and the Poincaré scalar field has been replaced by $\phi^{-8/w}$. The derivatives \mathcal{D}_μ are Lorentz covariant. Some of the (supercovariant) equations of motion are

$$\hat{\mathcal{D}}_a(\phi^{-1} \hat{\mathcal{D}}_a \phi) + \frac{2}{3} 7! w \phi^{-12/w} \hat{R}(A) \cdot \hat{R}(A) + \frac{1}{2} w \bar{\lambda} \Gamma^{\mu\nu} \hat{R}_{\mu\nu}(\psi) = 0 , \tag{5.2}$$

$$\hat{R}(e) + \frac{36}{w^2} (\phi^{-1} \hat{\mathcal{D}}_\mu \phi)^2 - 2 \cdot 7! \phi^{-12/w} \hat{R}(A) \cdot \hat{R}(A) = 0 , \tag{5.3}$$

$$\Gamma^\mu \hat{\mathcal{D}}_\mu \lambda = 0 , \tag{5.4}$$

$$\Gamma_\mu{}^{\rho\sigma} \hat{R}_{\rho\sigma}(\psi) - \frac{36}{w} (\phi^{-1} \hat{\mathcal{D}}_\rho \phi) \Gamma^\rho \Gamma_\mu \lambda - 6 \Gamma^{(7)} \Gamma_\mu \lambda \hat{R}(A)_{(7)} \phi^{-6/w} = 0 . \tag{5.5}$$

The caret indicates covariantization with respect to the gauge symmetries of Poincaré supergravity; $R(e)$ is the Ricci scalar and $R(\psi)$ the Rarita–Schwinger field strength $R_{\mu\nu}(\psi) = \partial_{[\mu}\psi_{\nu]}$.

Let us now make the relation between (5.2)–(5.5) and the superconformal algebra explicit. We make the gauge choice $b_\mu = 0$. As one can see from (3.29) the Q (and S) transformations must be modified by a compensating K transformation. The new Q transformation is

$$\delta'_Q = \delta_Q + \delta_K(\Lambda_K^a = -\frac{1}{2}\bar{\epsilon}\phi_a). \quad (5.6)$$

However, the independent fields (except b_μ) are inert under K transformations, so that the compensating transformation does not modify (3.34). Since the commutator of a Q and a K transformation involves only S , the modification in the algebra takes the form of a field-dependent S transformation:

$$[\delta'_Q(\varepsilon_1), \delta'_Q(\varepsilon_2)] = \text{as in (3.37)} \\ + \delta_S(\frac{1}{32}\bar{\varepsilon}_2\Gamma^a\varepsilon_1\Gamma^\mu\Gamma_a\phi_\mu + \frac{1}{7680}\bar{\varepsilon}_2\Gamma^{(5)}\varepsilon_1\Gamma^\mu\Gamma_{(5)}\phi_\mu). \quad (5.7)$$

To see the relation with the Poincaré theory we must solve ϕ_μ from the conventional constraint (3.31). One finds

$$\phi_\mu = \frac{1}{8}\Gamma_\mu^{\rho\sigma}\hat{R}_{\rho\sigma}(\psi) - \frac{1}{9}\Gamma_\mu\Gamma^{\rho\sigma}\hat{R}_{\rho\sigma}(\psi). \quad (5.8)$$

Substituting (5.8), the parameter of the field-dependent S transformation in (5.7) becomes

$$\eta = \left[\frac{1}{128}\bar{\varepsilon}_2\Gamma^\lambda\varepsilon_1(\frac{5}{6}\Gamma_\lambda\Gamma^\mu - 7\delta_\lambda^\mu) + \frac{1}{64 \cdot 1280}\bar{\varepsilon}_2\Gamma^{(5)}\varepsilon_1\Gamma^\mu\Gamma_{(5)} \right] \\ \times \left(\Gamma_\mu^{\rho\sigma}\hat{R}_{\rho\sigma}(\psi) - \frac{36}{w}(\phi^{-1}\hat{\mathcal{D}}_\rho\phi)\Gamma^\rho\Gamma_\mu\lambda - 6\Gamma^{(7)}\Gamma_\mu\lambda\phi^{-6/w}\hat{R}(A)_{(7)} \right), \quad (5.9)$$

which corresponds to the gravitino equation of motion (5.5).

In a similar way one can work out the form of the constraints (3.35) and (3.36) after the gauge choice $b_\mu = 0$ and the substitution of ϕ_μ and f_μ^a . One needs in particular f_μ^μ , which is given by

$$f_\mu^\mu = \frac{1}{9}\hat{R}(e) - \frac{1}{2}\bar{\psi}^\mu\phi_\mu. \quad (5.10)$$

The resulting equations are

$$\Gamma^\mu\hat{\mathcal{D}}_\mu\lambda + \frac{1}{9}\left(\Gamma^{\rho\sigma}\hat{R}_{\rho\sigma}(\psi) + \frac{36}{w}\Gamma^\mu(\phi^{-1}\hat{\mathcal{D}}_\mu\phi)\lambda - 3\Gamma^{(7)}\lambda\phi^{-6/w}\hat{R}(A)_{(7)}\right) = 0, \quad (5.11)$$

$$\hat{\mathcal{D}}_a(\phi^{-1}\hat{\mathcal{D}}_a\phi) + \frac{3}{2}w\bar{\lambda}\Gamma^{(7)}\lambda\phi^{-6/w}\hat{R}(A)_{(7)} + 2240w\phi^{-12/w}\hat{R}(A) \cdot \hat{R}(A) \\ + \frac{1}{9}w\left(\hat{R}(e) + \frac{36}{w^2}(\phi^{-1}\hat{\mathcal{D}}_a\phi)^2\right) = 0, \quad (5.12)$$

which correspond to Poincaré equations of motion. To show this correspondence we did not need the equation of motion of $A_{\mu_1 \dots \mu_6}$, nor the uncontracted Einstein equations. Of course these equations must hold as well, since they follow from the supersymmetry variation of the gravitino field equation (5.5).

The conformal multiplet contains the same fields as the Poincaré multiplet, but λ and ϕ satisfy constraints. It appears therefore that the conformal multiplet does not, after all, contain all the fields of Poincaré supergravity. This apparent discrepancy was noted previously in $d = 5$ [10] and $d = 10$ [8, 15] in the analysis of the supercurrents. On the other hand, the conformal multiplet corresponds to a massive representation of the $d = 10$ super-Poincaré algebra, which should decompose into massless supermultiplets if suitable covariant restrictions are imposed on the fields. One of these massless multiplets contains the same states as Poincaré supergravity; therefore it should be possible to give the explicit restrictions which lead to the on-shell Poincaré theory. These conditions are not those one might naively expect; as we shall see, the Rarita-Schwinger equation on the S invariant combination $\psi_\mu + \Gamma_\mu \lambda$ is too strong a restriction and sets the multiplet equal to zero. We will obtain the correct covariant conditions by combining information from the decomposition of massive into massless multiplets with supersymmetry.

The decomposition of the massive representations contained in the superconformal multiplet into massless representations is

$$\begin{aligned} \{44\} &\rightarrow \{35\} + \{8\} + \{1\}, \\ \{84\} &\rightarrow \{28\} + \{56\}, \\ \{128\} &\rightarrow \{56\} + \{8\} + \{56\} + \{8\}, \end{aligned}$$

for the spin-2, the 6-index antisymmetric tensor and the spin- $\frac{3}{2}$ representation respectively. In the on-shell Poincaré theory the antisymmetric tensor describes 28 massless states. This is a consequence of the Poincaré equation of motion

$$\partial^\mu R(A)_{\mu\lambda_1 \dots \lambda_6} = 0. \quad (5.13)$$

We will impose the condition (5.13) on the superconformal multiplet. Supersymmetry variations of (5.13) will imply further conditions, and together these should restrict the superconformal multiplet to the on-shell Poincaré fields.

To determine the consequences of (5.13) we consider the linearized conformal transformation rules (3.20). We impose the D and S gauge choices $\phi = 1$ and $\lambda = 0$. These gauge choices require compensating transformations, which lead to the following transformation rules:

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta \psi_\mu &= \mathcal{D}_\mu \epsilon + \frac{1}{6} (\Gamma_\mu \Gamma^{(7)} - 3 \Gamma^{(7)} \Gamma_\mu) \epsilon R(A)_{(7)}, \\ \delta A_{\mu_1 \dots \mu_6} &= \frac{3}{4 \cdot 6!} \bar{\epsilon} \Gamma_{[\mu_1 \dots \mu_5} \psi_{\mu_6]}. \end{aligned} \quad (5.14)$$

The linearized constraints (3.21) take on the form

$$\Gamma^{\mu\nu}\partial_\mu\psi_\nu = 0, \quad R(e) = 0. \quad (5.15)$$

Supersymmetry variations of (5.13) combined with the constraints (5.15) now lead to the following conditions on ψ_μ and $e_\mu{}^a$:

$$\not{\partial}\partial_{[\mu}\psi_{\nu]} = 0, \quad (5.16)$$

$$\partial_{[\mu}R_{\nu]}{}^a(e) = 0, \quad (5.17)$$

where $R_{\mu\nu}(e)$ is the Ricci-tensor. In a suitable gauge (5.16) implies $\square\psi_\mu = 0$, so that one obtains massless representations, but it does not imply the Rarita-Schwinger equation. A careful analysis of the conditions (5.15)–(5.17) shows that they indeed describe massless states in the representations $\{56\} + \{8\}$ corresponding to ψ_μ , and $\{35\} + \{1\}$ for $e_\mu{}^a$. The technical details of this analysis are gathered in appendix B.

The transformations (5.14) show that the Rarita-Schwinger equation for ψ_μ would be too strong a condition. One finds that the Rarita-Schwinger equation implies that $R(A)$, and thus the whole multiplet, vanishes. The transformations (5.14) should be compared with the supersymmetry variation of ψ_μ in $d = 10$ Poincaré supergravity. There the two terms containing $R(A)$ have a relative coefficient -2 , which is necessary and sufficient to ensure that the Rarita-Schwinger equation implies only (5.13).

Now that we have clarified the relation between (on-shell) Poincaré and (off-shell) Weyl supergravity, we may also compare their respective coupling to supersymmetric matter. The obvious case to consider is $d = 10$ supersymmetric Yang-Mills theory. In fact it has been noted previously that the action for the abelian gauge theory when coupled to supergravity was locally scale invariant, and a relation with conformal supergravity was suggested [15]. To elucidate this let us first consider the coupling of the supersymmetric Yang-Mills theory to conformal supergravity.

The $d = 10$ super-Yang-Mills theory contains the fields A_μ and χ , which both belong to the adjoint representation of the Yang-Mills group G . The lagrangian is [13]

$$\mathcal{L} = g^{-2} \text{tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \bar{\chi} \Gamma^\mu \mathcal{D}_\mu \chi \right], \quad (5.18)$$

where the Yang-Mills field strength and covariant derivative are defined by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \\ \mathcal{D}_\mu \chi &= \partial_\mu \chi - [A_\mu, \chi]. \end{aligned} \quad (5.19)$$

The first step in the superconformal coupling is to assign Weyl weights to A_μ and χ . An obvious choice is

$$w(A_\mu) = 0, \quad w(\chi) = \frac{3}{2}. \quad (5.20)$$

To lowest order, the transformation rules can then be written as

$$\begin{aligned}\delta A_\mu &= \frac{1}{2}\bar{\epsilon}\Gamma_\mu\chi, \\ \delta\chi &= -\frac{1}{4}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon,\end{aligned}\tag{5.21}$$

without the need for modifications with powers of the would-be compensator field ϕ . The scale-invariant action, which is the starting point of the Noether procedure, is given by

$$\mathcal{L} = g^{-2}e\phi^{6/w} \text{tr} \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\chi}\Gamma^\mu\mathcal{D}_\mu\chi \right],\tag{5.22}$$

with g the Yang–Mills coupling constant. Since we are considering the coupling of an on-shell theory ($d = 10$ Yang–Mills) to an off-shell supergravity multiplet ($d = 10$ conformal supergravity) the transformation rules of the supergravity fields should remain unchanged. This means that there are now two ways to proceed from the starting point (5.21) and (5.22). On the one hand, one can apply the familiar Noether method, which leads to a calculation quite similar to the one given in [15]. On the other hand, one can impose the algebra (3.37) on the fields A_μ and χ , which will require modifications of (5.21). On the field χ the algebra will only close modulo the χ equation of motion. Therefore, as one proceeds with the modifications of (5.21), one will unambiguously uncover the modifications to the χ equation of motion as well. At the end of this iterative procedure the Q variation of the χ equation of motion will yield the equation of motion of the Yang–Mills gauge field, and the complete action can be reconstructed from there.

Since A_μ does not transform under Lorentz and S transformations, the commutator on A_μ is particularly simple. One does not need to modify the transformation of A_μ itself, but the fully covariant translation and the λ -dependent Q transformation can be obtained by adding terms to $\delta\chi$. The resulting variation is

$$\delta\chi = -\frac{1}{4}\Gamma^{\mu\nu}\hat{F}_{\mu\nu}\epsilon + \frac{21}{32}\epsilon(\bar{\lambda}\chi) - \frac{21}{64}\Gamma^{(2)}\epsilon(\bar{\lambda}\Gamma_{(2)}\chi) - \frac{1}{256}\Gamma^{(4)}\epsilon(\bar{\lambda}\Gamma_{(4)}\chi),\tag{5.23}$$

where the superconformal Yang–Mills field strength is

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \bar{\psi}_{[\mu}\Gamma_{\nu]}\chi.\tag{5.24}$$

One now checks that the $\{Q, S\}$ commutator on χ closes as required. After this small triumph, it remains to be checked that the $\{Q, Q\}$ commutator closes as well. This is indeed the case, modulo an expression that can be identified with the χ equation of motion

$$\phi^{-3/w}\mathcal{D}(\phi^{3/w}\chi) + \Gamma^{(7)}\chi R(A)_{(7)}\phi^{-6/w} + \frac{3}{2}\Gamma\cdot\hat{F}\lambda + \frac{9}{32}\Gamma^{(3)}\chi\bar{\lambda}\Gamma_{(3)}\lambda = 0.\tag{5.25}$$

The derivative D_μ in (5.25) is again covariant with respect to all gauge symmetries. The Q variation of (5.25) leads to

$$\begin{aligned}D^b(\phi^{6/w}\hat{F}_{ba}) + 3\bar{\lambda}\Gamma_{ab}D^b(\phi^{6/w}\chi) + 3\phi^{6/w}\bar{\chi}D_a\lambda \\ - 2i\epsilon^{ab_1\cdots b_9}\hat{F}_{b_1b_2}R(A)_{b_3\cdots b_9} = 0.\end{aligned}\tag{5.26}$$

These equations of motion follow from the following action:

$$\begin{aligned} \mathcal{L} = & g^{-2} e \phi^{6/w} \operatorname{tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \bar{\chi} \Gamma^\mu \mathcal{D}_\mu (\omega, b, A) \chi \right. \\ & + \frac{1}{8} \bar{\chi} \Gamma^\mu \Gamma^{\nu\rho} (F_{\nu\rho} + \hat{F}_{\nu\rho}) (\psi_\mu + \Gamma_\mu \lambda) \\ & + \frac{1}{128} \bar{\chi} \Gamma^{(3)} \chi \bar{\psi}_\mu (4 \Gamma^{(3)} \Gamma^\mu + 3 \Gamma^\mu \Gamma^{(3)}) \lambda + \frac{9}{64} \bar{\chi} \Gamma^{(3)} \chi \bar{\lambda} \Gamma_{(3)} \lambda \left. \right] \\ & + \frac{1}{2} i g^{-2} \varepsilon^{\mu_1 \dots \mu_{10}} \operatorname{tr} [F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} A_{\mu_5 \dots \mu_{10}} + \frac{1}{2} g^{-2} e \operatorname{tr} [\bar{\chi} \Gamma^{(7)} \chi] R(A)_{(7)}] . \end{aligned} \quad (5.27)$$

Of course one can also directly verify that (5.27) is invariant under all superconformal transformations. For establishing the invariance under Q transformations one needs the constraint (3.35). This constraint is also relevant in order to ensure that the gauge fields of conformal supergravity, ψ_μ and $h_{\mu\nu}$, couple to the (gamma-)traceless part of the currents J_μ and $\theta_{\mu\nu}$. This fact, which we have generally discussed in sect. 3, can now be verified explicitly from the coupling terms contained in (5.26). To lowest order in the supergravity fields the Noether term is given by

$$\mathcal{L}_N = \operatorname{tr} \left[\frac{1}{4} \bar{\chi} \Gamma^\mu \Gamma \cdot F (\psi_\mu + \Gamma_\mu \lambda) \right] . \quad (5.28)$$

Due to the constraint (3.15) this can be written in the form

$$\bar{\psi}_\mu \operatorname{tr} \left[\frac{1}{4} \Gamma \cdot F \Gamma^\mu \chi + \frac{1}{6} \left(\delta_{\mu\nu} - \frac{1}{\square} \partial_\mu \partial_\nu \right) \Gamma_\nu \Gamma \cdot F \chi \right] , \quad (5.29)$$

so that the current \hat{J}_μ that couples to the gauge field ψ_μ is traceless: $\Gamma \cdot \hat{J} = 0$. In a similar fashion one shows that the gravitational field $h_{\mu\nu}$ couples to the traceless energy momentum tensor $\hat{\theta}_{\mu\nu}$.

Since the Q -supersymmetry transformations of the superconformal fields coincide with the supersymmetry variations of on-shell Poincaré supergravity, there must be a close relationship between our results and those of Yang–Mills–Einstein supergravity. Chamseddine [14] has given the action and transformation rules for the latter up to quartic fermionic terms. Even although our results have been obtained in an entirely different context, (5.27) is in complete agreement with his results. Similar results exist for Yang–Mills–Einstein supergravity in the formulation based on a two-index gauge field [18]. The two versions are related by a duality transformation. For the abelian theory this was demonstrated in [15], and we briefly indicate its non-abelian extension here. After combining the lagrangians (5.1) and (5.27) we write $F_{[\mu\nu} F_{\rho\sigma]}$ as the derivative of the Chern–Simons form

$$\operatorname{tr} F_{[\mu\nu} F_{\rho\sigma]} = \frac{1}{2} \partial_{[\mu} \operatorname{tr} [A_\nu F_{\rho\sigma]} + \frac{2}{3} A_\nu A_\rho A_{\sigma]} . \quad (5.30)$$

After a partial integration the dependence on the six-index gauge field is now

contained in its field strength $R^0(A)$. We give the relevant terms:

$$\begin{aligned}\mathcal{L}_{\text{YME}} = & -4 \cdot 7! e \phi^{-12/w} R^0(A)_{(7)} R^0(A)^{(7)} \\ & + \frac{1}{2} e \phi^{-6/w} R^0(A)_{(7)} (\bar{\psi}_{[\mu} \Gamma^\mu \Gamma^{(7)} \Gamma^\nu \psi_{\nu]} + 12 \bar{\psi}_\mu \Gamma^{(7)} \Gamma^\mu \lambda) \\ & + \frac{1}{2} g^{-2} R^0(A)_{\mu_1 \dots \mu_7} \text{tr} [e \bar{\chi} \Gamma^{\mu_1 \dots \mu_7} \chi + \frac{1}{2} i \varepsilon^{\mu_1 \dots \mu_{10}} (A_{\mu_8} F_{\mu_9 \mu_{10}} + \frac{2}{3} A_{\mu_8} A_{\mu_9} A_{\mu_{10}})] \\ & + R^0(A)\text{-independent terms}.\end{aligned}\quad (5.31)$$

The action corresponding to (5.31) is equivalent to

$$\begin{aligned}\mathcal{L}' = & \frac{3}{4} e \phi^{-12/w} (t_{\mu\nu\rho})^2 - \frac{3}{2} e t^{\mu\nu\rho} \partial_\mu A_{\nu\rho} \\ & - e \frac{1}{16} \sqrt{2} \phi^{-6/w} t_{\rho\sigma\tau} (\bar{\psi}_{[\mu} \Gamma^\mu \Gamma^{\rho\sigma\tau} \Gamma^\nu \psi_{\nu]} - 12 \bar{\psi}_\mu \Gamma^{\rho\sigma\tau} \Gamma^\mu \lambda) \\ & - e g^{-2} \frac{1}{16} \sqrt{2} t_{\mu\nu\rho} \text{tr} [\bar{\chi} \Gamma^{\mu\nu\rho} \chi] \\ & - e g^{-2} \frac{3}{16} \sqrt{2} t^{\mu\nu\rho} \text{tr} [A_\mu F_{\nu\rho} + \frac{2}{3} A_\mu A_\nu A_\rho] + \dots,\end{aligned}\quad (5.32)$$

where we have introduced two antisymmetric fields $A_{\mu\nu}$ and $t_{\mu\nu\rho}$. The first one is a gauge field, whose field equation implies that $t_{\mu\nu\rho}$ is divergence free: $\partial_\mu (e t^{\mu\nu\rho}) = 0$. Therefore $t_{\mu\nu\rho}$ can be written as

$$\begin{aligned}t^{\mu\nu\rho} = & \frac{2}{3} \sqrt{2} i e^{-1} \varepsilon^{\mu\nu\rho\sigma_1 \dots \sigma_7} \partial_{\sigma_1} A_{\sigma_2 \dots \sigma_7} \\ = & \frac{2}{3} \sqrt{2} i e^{-1} \varepsilon^{\mu\nu\rho\sigma_1 \dots \sigma_7} R^0(A)_{\sigma_1 \dots \sigma_7},\end{aligned}\quad (5.33)$$

whose substitution leads back to (5.31). Alternatively one can also solve the field equation for $t_{\mu\nu\rho}$; the result is

$$\begin{aligned}t_{\mu\nu\rho} = & \phi^{12/w} \partial_{[\mu} A_{\nu\rho]} + \frac{1}{24} \sqrt{2} \phi^{6/w} (\bar{\psi}_{[\sigma} \Gamma^\sigma \Gamma_{\mu\nu\rho} \Gamma^\tau \psi_{\tau]} - 12 \bar{\psi}_\tau \Gamma_{\mu\nu\rho} \Gamma^\tau \lambda) \\ & + g^{-2} \frac{1}{24} \sqrt{2} \phi^{12/w} \text{tr} [\bar{\chi} \Gamma_{\mu\nu\rho} \chi + 3 (A_{[\mu} F_{\nu\rho]} + \frac{2}{3} A_{[\mu} A_\nu A_{\rho]})].\end{aligned}\quad (5.34)$$

Substitution of (5.34) yields the results obtained in [14, 18]. Note that the invariance of (5.32) under the non-abelian gauge transformations requires that $A_{\mu\nu}$ is not inert; under these transformations one finds

$$\delta_{\text{YM}} A_{\mu\nu} = -\frac{1}{4} \sqrt{2} g^{-2} \text{tr} [\Lambda \partial_{[\mu} A_{\nu]}], \quad (5.35)$$

where Λ denotes the (Lie-algebra valued) infinitesimal parameter of the gauge transformation. A noteworthy feature of the transformations (5.35) is that they fail to close within the set of Yang–Mills transformations. Considering the commutator of two Yang–Mills transformations with parameters Λ_1 and Λ_2 one finds

$$[\delta_{\text{YM}}(\Lambda_1), \delta_{\text{YM}}(\Lambda_2)] A_{\mu\nu} = -\frac{1}{4} \sqrt{2} g^{-2} \{ \text{tr} [[\Lambda_2, \Lambda_1] \partial_{[\mu} A_{\nu]}] + \partial_{[\mu} \text{tr} [[\Lambda_2, \Lambda_1] A_{\nu}]] \}, \quad (5.36)$$

where the first term is implied by the structure of the Yang–Mills group, and the second one represents a tensor gauge transformation of $A_{\mu\nu}$.

Hence we have established that, although the lagrangian (5.27) has been constructed in the context of conformal supergravity with fields ϕ and λ subject to differential constraints, one can make direct contact with results derived for matter couplings in (on-shell) Poincaré supergravity. We should emphasize that the conformal theory uniquely selects a six-index gauge field. The conversion of a six-index into a two-index gauge field can only take place within the context of Poincaré supergravity.

6. Towards Poincaré supergravity

In the standard approach by which Poincaré supergravity is obtained from conformal supergravity one introduces a number of compensating supermultiplets. The resulting field representation is then shown to be gauge equivalent to an irreducible representation of Poincaré supergravity. In principle this procedure works in ten dimensions, except that one expects to be left with fields that are still subject to differential constraints. Hence, although one may systematically construct invariant actions for Poincaré supergravity, those will usually not be defined in terms of unrestricted fields.

On the other hand we may further explore the close relationship between the Poincaré and superconformal fields by rewriting the Poincaré supergravity action (5.1) using superconformal notions. This can be done directly by introducing two compensating fields, a scalar field A associated with dilatations and a chiral spinor ξ with S supersymmetry. We choose the following D and S transformations for A and ξ :

$$\begin{aligned}\delta A &= 8\Lambda_D A, \\ \delta \xi &= 4A\eta + \frac{17}{2}\Lambda_D \xi.\end{aligned}\tag{6.1}$$

The Weyl weight of A and the S supersymmetry variation are entirely a matter of choice whereas the Weyl weight of ξ follows from the $[D, S]$ commutator; the essential feature is that $\ln A$ and $A^{-1}\xi$ transform under D and S with an inhomogeneous term which is characteristic for a compensating field. The Poincaré fields are now written as D - and S -invariant combinations of the superconformal and compensating fields:

$$\begin{aligned}(e_\mu^a)^{\text{Poincaré}} &= A^{1/8} e_\mu^a, \\ (\psi_\mu)^{\text{Poincaré}} &= A^{1/16} \psi_\mu + \frac{1}{4} A^{-15/16} \Gamma_\mu \xi, \\ (A_{\mu_1 \dots \mu_6})^{\text{Poincaré}} &= A_{\mu_1 \dots \mu_6}, \\ (\lambda)^{\text{Poincaré}} &= A^{-1/16} \lambda - \frac{1}{4} A^{-17/16} \xi, \\ (\phi)^{\text{Poincaré}} &= A^{-w/8} \phi.\end{aligned}\tag{6.2}$$

The Poincaré transformations of the Poincaré fields on the left-hand side of (6.2)

can now be compared to the superconformal transformations acting on the right-hand side of this equation. In this way one identifies all superconformal transformations of the compensating fields. In particular the Q -supersymmetry transformations are

$$\begin{aligned}\delta A &= \bar{\epsilon} \xi, \\ \delta \xi &= \frac{1}{4} \not{D} A \epsilon + \frac{5}{64} \Gamma^{(3)} \epsilon \bar{\xi} \Gamma_{(3)} \lambda \\ &\quad + \frac{21}{32} \Gamma^a \epsilon \bar{\lambda} \Gamma_a \xi - \frac{1}{2560} \Gamma^{(5)} \epsilon \bar{\lambda} \Gamma_{(5)} \xi.\end{aligned}\quad (6.3)$$

If we now substitute (6.2) into the Poincaré supergravity lagrangian (5.1) then the corresponding action is expressed in terms of superconformal fields and two compensators and it is formally invariant under all the superconformal symmetries. Explicit substitution thus leads to the following supergravity lagrangian:

$$e^{-1} \mathcal{L} = AC + \bar{\xi} \Psi - \frac{1}{4} \bar{\Psi} \Gamma^\mu A \Psi, \quad (6.4)$$

where C and Ψ are defined by

$$\begin{aligned}C &= \frac{1}{4} \phi^{-4/w} \square \phi^{4/w} + 4 \bar{\lambda} \phi^{-4/w} \not{D} (\phi^{4/w} \lambda) + 2240 \phi^{-12/w} (R(A))^2, \\ \Psi &= -2 \phi^{-4/w} \not{D} (\phi^{4/w} \lambda) + \frac{2}{3} \Gamma^{(7)} \lambda R(A)_{(7)} \phi^{-6/w}.\end{aligned}\quad (6.5)$$

These expressions, which depend only on the superconformal fields, turn out to correspond to linear combinations of the differential constraints (3.35), (3.36) which must be imposed in order that the superconformal fields define an off-shell field representation. However, for the moment we are not basing ourselves on such a representation, and we are essentially only rewriting the Poincaré theory. We should add that the invariance of (6.4) can, of course, also be verified independently by direct application of the transformation rules (3.34) combined with (6.3). One then needs the transformation rules of C and Ψ that are induced by (3.34) [simply ignoring the constraints (3.35), (3.36)], which take the form

$$\begin{aligned}\delta \Psi &= -C \epsilon - \frac{51}{16} \epsilon (\bar{\lambda} \Psi) - \frac{9}{32} \Gamma^{(2)} \epsilon (\bar{\lambda} \Gamma_{(2)} \Psi) \\ &\quad + \frac{1}{128} \Gamma^{(4)} \epsilon (\bar{\lambda} \Gamma_{(4)} \Psi), \\ \delta C &= -\frac{1}{4} \bar{\epsilon} \not{D} \Psi - \frac{1}{8} \bar{\epsilon} \Gamma^{(7)} \Psi \phi^{-6/w} R(A)_{(7)} \\ &\quad + \frac{9}{128} \bar{\epsilon} \Gamma_{(3)} \Psi \bar{\lambda} \Gamma^{(3)} \lambda - \frac{3}{2} \bar{\eta} \Psi.\end{aligned}\quad (6.6)$$

The lagrangian (6.4) is linear in the compensating fields A and ξ . Therefore the equations of motion of A and ξ imply that Ψ and C vanish on-shell. This should not come as a surprise because we have already seen in sect. 5 that Ψ and C are intimately related to Poincaré field equations, and (6.4) follows directly from (5.1).

To proceed beyond the on-shell Poincaré theory and (6.4) we must first extend the compensating fields A and ξ to a complete multiplet of Lagrange multipliers, and correspondingly we must also extend the multiplet based on the combinations

(6.5). We first note that A and ξ form the beginning of a scalar (chiral) multiplet. It is in principle straightforward to determine the superconformal transformation rules of a scalar multiplet in a superconformal background. Its first few components are a scalar A , a chiral spinor ξ , an antisymmetric tensor T_{abc} and an antisymmetric spinor Λ_{ab} which transform under Q and S supersymmetry as

$$\begin{aligned}
 \delta A &= \bar{\epsilon} \xi, \\
 \delta \xi &= \frac{1}{4} \not{D} A \epsilon + \Gamma^{(3)} \epsilon T_{(3)} + \frac{21}{32} \Gamma_a \bar{\epsilon} \Gamma^a \xi + \frac{5}{64} \Gamma^{(3)} \epsilon \bar{\epsilon} \Gamma_{(3)} \xi \\
 &\quad - \frac{1}{2560} \Gamma^{(5)} \epsilon \bar{\epsilon} \Gamma_{(5)} \xi + \frac{1}{2} w_0 A \eta, \\
 \delta T_{abc} &= -\frac{1}{192} \bar{\epsilon} \not{D} \Gamma_{abc} \xi + \bar{\epsilon} \Gamma_{[a} \Lambda_{bc]} \\
 &\quad + \frac{1}{192} \bar{\epsilon} \left(\frac{9}{2} (\not{D} A) \Gamma_{abc} \lambda + \Gamma_{abc} (\not{D} A) \lambda \right) \\
 &\quad - \frac{w_0}{192 w} \bar{\epsilon} \left(\frac{9}{2} (\phi^{-1} \not{D} \phi) \Gamma_{abc} A \lambda + \Gamma_{abc} (\phi^{-1} \not{D} \phi) A \lambda \right) \\
 &\quad + \frac{1}{96 w} \bar{\epsilon} \left(\frac{9}{2} (\phi^{-1} \not{D} \phi) \Gamma_{abc} \xi + \Gamma_{abc} (\phi^{-1} \not{D} \phi) \xi \right) + \dots \\
 &\quad + \frac{1}{96} (w_0 - 8) \bar{\eta} \Gamma_{abc} \xi,
 \end{aligned} \tag{6.7}$$

where w_0 is the Weyl weight of the multiplet, and Λ_{ab} satisfies $\Gamma^a \Lambda_{ab} = 0$.

Our arguments indicate that one needs at least a scalar multiplet of Lagrange multipliers. Since the lagrangian (6.4) must retain its structure when we include further components of the multiplier multiplet we conclude that also the expressions (6.5) should somehow generalize to at least a scalar multiplet*. At this point we realize that this multiplet is the supersymmetric extension of the constraint equations of conformal supergravity. To include the full multiplet therefore corresponds to relaxing the constraints in order to obtain an unrestricted field representation consisting of the superconformal fields combined with an extra submultiplet. When the latter is put to zero we recover pure conformal supergravity as presented in sect. 3.

It is in principle straightforward to extend the field representation according to the above arguments and find an off-shell formulation for Poincaré supergravity. If one assumes that the submultiplet is precisely a scalar multiplet then the representation coincides with the fields that couple to the Yang–Mills supercurrent. Consequently one should be able to make contact with the off-shell version of linearized Poincaré supergravity obtained recently by Howe et al. [9]. The lagrangian (6.4) then generalizes as the product of two scalar multiplets

$$\mathcal{L} = \int d^{16} \theta S(x, \theta) \Phi(x, \theta), \tag{6.8}$$

* We note that this observation is consistent with the fact that on-shell Poincaré supergravity is described by a scalar multiplet as well [19].

with proper non-linear modifications, where $S(x, \theta)$ is the multiplet of Lagrange multipliers and $\Phi(x, \theta)$ the multiplet ending with Ψ and C . In components we thus obtain

$$e^{-1}\mathcal{L} = AC + \bar{\xi}\Psi + T_{abc}u^{abc} + \bar{\Lambda}_{ab}\chi^{ab} + \dots \\ - \frac{1}{4}\bar{\psi}_\mu\Gamma^\mu(A\Psi + \frac{1}{48}\Gamma^{(3)}u_{(3)}\xi + \dots) + \dots, \quad (6.9)$$

where u and χ are the fields belonging to Φ that are conjugate to T and Λ . In (6.9) C and Ψ are no longer given by (6.5) and one should be prepared to include further modifications depending on the lower dimensional components of Φ and the superconformal fields.

Before giving some of these modifications we must first relax the constraint $\Phi(x, \theta) = 0$ in the superconformal field representation. Previously the superconformal algebra closed on all the fields except ψ_μ , where one had to rely on the constraint $\Psi = 0$. Therefore, ignoring the constraint forces us to introduce a field of $\Phi(x, \theta)$ into the Q variation of ψ_μ , which must transform into Ψ . Inspection of the scalar multiplet shows that this field must be u_{abc} , which varies into Ψ according to

$$\delta u_{abc} = -\bar{\epsilon}\Gamma_{abc}\Psi. \quad (6.10)$$

The corresponding variation of ψ_μ which leads again to a closed algebra as far as terms proportional to Ψ are concerned is now uniquely given:

$$\delta^{\text{new}}\psi_\mu = \frac{1}{384}(4\Gamma_\mu\Gamma^{abc} + 9\Gamma^{abc}\Gamma_\mu)\epsilon u_{abc}. \quad (6.11)$$

It should come as no surprise that (6.11) coincides with the linearized transformations that follow from the Yang–Mills supercurrent [15]. The field u_{abc} couples to the Yang–Mills current $\text{tr}(\bar{\chi}\Gamma^{abc}\chi)$.

Of course, there are many new terms in the transformation rules. We have determined the following contributions:

$$\delta^{\text{new}}\psi_\mu = \frac{1}{384}(4\Gamma_\mu\Gamma^{(3)} + 9\Gamma^{(3)}\Gamma_\mu)\epsilon u_{(3)} - \frac{3}{5120}\bar{\epsilon}\Gamma^{a_1\dots a_5}\psi_\mu\Gamma_{a_1\dots a_3}\chi_{a_4a_5}, \\ \delta^{\text{new}}A_{\mu_1\dots\mu_6} = -\frac{1}{1024}\phi^{6/w}\bar{\epsilon}\Gamma_{[\mu_1\dots\mu_4}\chi_{\mu_5\mu_6]}, \\ \delta\chi_{ab} = -\frac{1}{72}(\Gamma_{ab}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{ab} + 3\Gamma_{[a}\Gamma^{(3)}\Gamma_{b]})\epsilon u_{(3)}, \\ \delta u_{abc} = -\bar{\epsilon}\Gamma_{abc}\Psi + \frac{3}{8}\bar{\epsilon}\not{D}\Gamma_{[a}\chi_{bc]}. \quad (6.12)$$

To find (6.12) one must take into account that the algebra of superconformal gauge transformations is now also modified by terms depending on the components of Φ . We have established the presence of the modifications

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] \rightarrow \delta_M(\frac{1}{384}\bar{\epsilon}_2(2\Gamma^{ab}\Gamma^{(3)} + 2\Gamma^{(3)}\Gamma^{ab} + 9\Gamma^{[a}\Gamma^{(3)}\Gamma^{b]})\epsilon_1 u_{(3)}) \\ + \delta_Q(-\frac{3}{5120}\bar{\epsilon}_2\Gamma^{a_1\dots a_5}\epsilon_1\Gamma_{a_1\dots a_3}\chi_{a_4a_5}). \quad (6.13)$$

This in turn implies that the transformations (6.7) of the scalar multiplet acquire extra terms in order to implement the modified algebra. We record some of the additional terms:

$$\begin{aligned}\delta\xi^{\text{new}} &= -\frac{3}{10240}\Gamma^{a_1\cdots a_5}\varepsilon\bar{\xi}\Gamma_{a_1\cdots a_3}\chi_{a_4a_5}, \\ \delta T_{abc}^{\text{new}} &= \frac{1}{4096}(\bar{\varepsilon}\Gamma_{abc}\Gamma^{(3)}\xi u_{(3)} + \frac{12}{5}\bar{\varepsilon}\Gamma_{[ab}\Gamma^{(3)}\Gamma_{c]}\xi u_{(3)}) \\ &\quad - \frac{47}{3840}\bar{\varepsilon}\xi u_{abc} + \frac{3}{64\cdot 2048}\bar{\lambda}\Gamma_{abc}\Gamma^{a_1\cdots a_5}\varepsilon\bar{\xi}\Gamma_{a_1\cdots a_3}\chi_{a_4a_5}.\end{aligned}\quad (6.14)$$

Taking into account the new variations (6.12), (6.14) it is then possible to explicitly obtain additional contributions to the lagrangian (6.9). Requiring invariance one finds that the extra terms can be accounted for by the following redefinition of C and Ψ :

$$\begin{aligned}C^{\text{new}} &= C^{\text{old}} - \frac{1}{384}(u_{abc})^2 + \cdots, \\ \Psi^{\text{new}} &= \Psi^{\text{old}} + \frac{19}{15360}\Gamma^{(2)}\Gamma^{(3)}\chi_{(2)}u_{(3)} + \cdots.\end{aligned}\quad (6.15)$$

Note in particular the absence of $\tilde{R}(A) \cdot u$ modifications in C and λu terms in Ψ .

The lagrangian (6.9) with the substitutions (6.15) remains gauge equivalent to the Poincaré supergravity lagrangian, where we have now also included some of the auxiliary fields. This result may be compared to that of [9], for instance after imposing the gauge conditions $A = 1$, $\xi = 0$. Of course, the structure of the lagrangian is quite different from the one presented in [9]. In the superconformal framework the kinetic terms of the Poincaré fields are implicitly contained as covariantizations in C and Ψ . Therefore, we do not need an additional term in the action to generate the kinetic terms. To make the correspondence more precise we remind the reader that in [9] the Poincaré fields were described by an unrestricted superfield V_{abc} subject to certain gauge transformations. The highest dimensional component of V_{abc} can be identified with u_{abc} , whereas the scalar submultiplet Φ corresponds to $\tilde{D}\Gamma^{abc}DV_{abc}$.

Hence we have presented a new method for constructing Poincaré from conformal supergravity. Of course, a full completion of our results along the lines above is extremely difficult in view of the large number of fields. Therefore, it may be useful to recast some of our methods in superspace form, before attempting a full determination of the theory.

We are grateful to A. Van Proeyen for stimulating and informative discussions. For two of us (E.B., B.d.W.) this work is part of the research programme of the “Stichting voor Fundamenteel Onderzoek der Materie” (F.O.M.).

Appendix A

GAMMA MATRICES IN TEN DIMENSIONS

The 32×32 Dirac matrices Γ_a ($a = 1, \dots, 10$) are defined by the property

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \quad (\text{A.1})$$

In this appendix we shall gather a number of useful properties of the Γ -matrices.

The completely antisymmetric product of n Γ -matrices is denoted by

$$\Gamma_{a_1 \dots a_n} \equiv \Gamma_{[a_1} \Gamma_{a_2} \dots \Gamma_{a_n]} \equiv (1/n!) \sum_P (-1)^P \Gamma_{a_1} \Gamma_{a_2} \dots \Gamma_{a_n}. \quad (\text{A.2})$$

When the index-structure is obvious from the context we often write $\Gamma_{(n)} = \Gamma_{a_1 \dots a_n}$. The set of matrices O_I ,

$$O_I = \{1, \Gamma_{(1)}, i\Gamma_{(2)}, i\Gamma_{(3)}, \Gamma_{(4)}, \dots, i\Gamma_{(10)}\}, \quad (\text{A.3})$$

forms a complete set and satisfies

$$\text{tr } O_I O_J = 32 \delta_{IJ}. \quad (\text{A.4})$$

This leads to the Fierz rearrangement formula

$$\psi \bar{\lambda} = -\frac{1}{32} \sum_{n=0}^5 (a_n/n!) \Gamma^{(n)} (\bar{\lambda} \Gamma_{(n)} \psi), \quad (\text{A.5})$$

with $a_0 = a_1 = a_4 = 2$, $a_2 = a_3 = -2$, $a_5 = 1$.

Throughout this paper we use spinors which are both Majorana ($\bar{\lambda} = \lambda^T C$) and chiral ($\Gamma_{11} \lambda = \pm \lambda$). The antisymmetric charge conjugation matrix C satisfies

$$(\Gamma_a)^T = -C \Gamma_a C^{-1}, \quad \text{for } a = 1, \dots, 11. \quad (\text{A.6})$$

In analogy to $d = 4$ we have defined $\Gamma_{11} = i\Gamma_{(10)}$, which has the properties

$$\Gamma_{11} \Gamma_{11} = 1, \quad \{\Gamma_{11}, \Gamma_a\} = 0. \quad (\text{A.7})$$

The introduction of Γ_{11} allows one to express $\Gamma_{(n)}$ in terms of $\Gamma_{(10-n)}$. This duality relation reads

$$\Gamma_{a_1 \dots a_n} = i(s_n/(10-n)!) \epsilon_{a_1 \dots a_n b_1 \dots b_{10-n}} \Gamma^{b_1 \dots b_{10-n}} \Gamma_{11}, \quad (\text{A.8})$$

with $s_n = +1$ for $n = 1, 4, 5, 8, 9$ and $s_n = -1$ for $n = 2, 3, 6, 7, 10$.

Products of Γ -matrices can be rewritten as a sum of independent terms. The basic multiplication is

$$\Gamma_{a_1 \dots a_n} \Gamma^b = \Gamma_{a_1 \dots a_n}{}^b + n \Gamma_{[a_1 \dots a_{n-1}} \delta_{a_n]}^b. \quad (\text{A.9})$$

Repeated use of (A.9) gives the result

$$\Gamma_{a_1 \dots a_n} \Gamma^{b_1 \dots b_m} = \sum_{k=0}^{\min(n,m)} (-1)^{k(2n+3-k)/2} \binom{n}{k} \binom{m}{k} k! \delta_{[a_1 \dots a_k}^{[b_1 \dots b_k} \Gamma_{a_{k+1} \dots a_n]}^{b_{k+1} \dots b_m]}, \quad (\text{A.10})$$

valid only for $m + n \leq 11$. If $m + n > 11$ one uses (A.8) for both $\Gamma_{(n)}$ and $\Gamma_{(m)}$ before applying (A.10).

One of the more frequently occurring calculations with Γ -matrices is the product

$$\Gamma_{a_1 \dots a_n} \Gamma^{b_1 \dots b_m} \Gamma^{a_1 \dots a_n} = n! c(n, m) \Gamma^{b_1 \dots b_m}. \quad (\text{A.11})$$

The coefficients $c(n, m)$ satisfy the recurrence relation

$$nc(n, m) = (-)^{n-1} c(1, m) c(n-1, m) + (12-n) c(n-2, m), \quad (\text{A.12})$$

and the identities

$$\begin{aligned} c(n, 10-m) &= (-)^n c(n, m), \\ c(10-n, m) &= (-)^{n+m+1} c(n, m). \end{aligned} \quad (\text{A.13})$$

Table 3 gives the coefficients $c(n, m)$ for $n, m < 5$. Eq. (A.13) then determines c for other values of n and m .

Appendix B

DEGREES OF FREEDOM OF ON-SHELL POINCARÉ SUPERGRAVITY

The conformal multiplet, in the form (5.14), can be subjected to the covariant conditions

$$\partial^\mu R(A)_{\mu\lambda_1 \dots \lambda_6} = 0, \quad (\text{B.1})$$

$$\partial_{[\mu} R_{\nu]\lambda}(e) = 0, \quad (\text{B.2})$$

$$\partial_{[\mu} \delta\psi_{\nu]} = 0. \quad (\text{B.3})$$

The conformal fields already satisfy the constraints

$$R(e) = 0, \quad (\text{B.4})$$

$$\Gamma^{\mu\nu} \partial_\mu \psi_\nu = 0. \quad (\text{B.5})$$

TABLE 3
The coefficients $c(n, m)$ defined in (A.11)

$m \backslash n$	0	1	2	3	4	5
0	1	10	-45	-120	210	252
1	1	-8	-27	48	42	0
2	1	6	-13	-8	-14	-28
3	1	-4	-3	-8	-14	0
4	1	2	3	8	2	12
5	1	0	5	0	10	0

The conditions (B.1)–(B.5) are consistent with the transformation rules (5.14). The degrees of freedom which remain when these conditions are imposed correspond to those of the on-shell Poincaré theory.

Consider first the spin-2 field of the conformal theory, which can be represented by a symmetric tensor $h_{\mu\nu}$ [cf. (3.11)]. In terms of $h_{\mu\nu}$ the Ricci tensor is

$$R_{\mu\nu}(e) = \frac{1}{2}(\square h_{\mu\nu} + \partial_\mu \partial_\nu h_{\rho\rho} - 2\partial_\rho \partial_{(\mu} h_{\nu)\rho}). \quad (\text{B.6})$$

We must show, that (B.2) and (B.4) restrict $h_{\mu\nu}$ to the degrees of freedom of massless spin-2 and spin-0. To do so it is convenient to go to momentum space, and to decompose each vector index in the independent vectors $p_\mu = (p, p_{10})$, $\bar{p}_\mu = (p, -p_{10})$ and the 8 transverse polarization vectors ε_μ^i , $i = 1, \dots, 8$. Of course $p \cdot \varepsilon^i = \bar{p} \cdot \varepsilon^i = 0$, but $p \cdot \bar{p} \neq 0$. In this decomposition $h_{\mu\nu}$ can be written as

$$\begin{aligned} h_{\mu\nu} = & a_{ij} \varepsilon_\mu^i \varepsilon_\nu^j + b_i \varepsilon_{(\mu}^i \bar{p}_{\nu)} + c \bar{p}_\mu \bar{p}_\nu \\ & + d_i \varepsilon_{(\mu}^i p_{\nu)} + e p_{(\mu} \bar{p}_{\nu)} + f p_\mu p_\nu. \end{aligned} \quad (\text{B.7})$$

The components along p_μ correspond to gauge degrees of freedom, and drop out of (B.2) and (B.4). Hence we can choose $d_i = e = f = 0$. The equations (B.2) and (B.4) now take on the form

$$\begin{aligned} p^2 p_{[\mu} (a_{ij} \varepsilon_{\nu]}^i \varepsilon_\lambda^j + \frac{1}{2} b_i (\varepsilon_{\nu]}^i \bar{p}_\lambda + \varepsilon_\lambda^i \bar{p}_{\nu]}) + c \bar{p}_{\nu]} \bar{p}_\lambda) \\ - p \cdot \bar{p} p_\lambda p_{[\mu} (\frac{1}{2} b_i \varepsilon_{\nu]}^i + c \bar{p}_{\nu]}) = 0, \end{aligned} \quad (\text{B.8})$$

$$p^2 (a_{ii} + c p^2) - (p \cdot \bar{p})^2 c = 0. \quad (\text{B.9})$$

From the independence of the tensorial structures in (B.8) we deduce that c and b_i must vanish. One then concludes that the remaining degrees of freedom are massless and represented by the symmetric matrix a_{ij} , which has $\frac{1}{2}(d-2)(d-1)$ elements. Its trace represents the spin-0 state and its traceless part the states of a spin-2 particle.

Had one imposed the Einstein equation, which implies that (B.6) vanishes, then the trace of a_{ij} would have vanished as well. This is caused by the second term in (B.6), which drops out in (B.2). So clearly the Einstein equation would be too strong a condition on the multiplet. In any case, it is not consistent with the transformation rules (5.14).

For the conformal gravitino one makes a decomposition similar to (B.7):

$$\psi_\mu = \varepsilon_\mu^i u_i + \bar{p}_\mu v + p_\mu w. \quad (\text{B.10})$$

Again the components along p_μ correspond to gauge degrees of freedom, and we can choose $w = 0$. Eqs. (B.3) and (B.5) take on the form

$$p_{[\mu} \varepsilon_{\nu]}^i \not{p} u_i + p_{[\mu} \bar{p}_{\nu]} \not{p} v = 0, \quad (\text{B.11})$$

$$\Gamma^{\mu\nu} (p_\mu \varepsilon_\nu^i u_i + p_\mu \bar{p}_\nu v) = 0. \quad (\text{B.12})$$

From the independent structures in (B.11) we conclude that

$$\not{p}u_i = \not{p}v = 0, \quad (\text{B.13})$$

and the constraint (B.12) then implies that

$$v = 0. \quad (\text{B.14})$$

So ψ_μ is now restricted to the 8 independent Majorana–Weyl spinors u_i , satisfying $\not{p}u_i = 0$. Using an explicit form of the Γ -matrices it is not difficult to show that this condition halves the number of independent components in each spinor (i.e. for each value of i) so that we have obtained 64 massless spinor degrees of freedom. One finally writes the 8 independent spinors in the form

$$s = \not{\epsilon}^i u_i, \quad (\text{B.15})$$

corresponding to the 8 degrees of freedom of a spin- $\frac{1}{2}$ particle, and

$$s_i = (\delta_i^j - \frac{1}{8}\not{\epsilon}_i \not{\epsilon}^j) u_j, \quad i = 1, \dots, 8, \quad (\text{B.16})$$

which, since $\not{\epsilon}^i s_i$ vanishes, represents the 56 degrees of freedom of a spin- $\frac{3}{2}$ particle. Note that the Rarita–Schwinger equation for ψ_μ implies that (B.15) vanishes as well, so that in that case only the spin- $\frac{3}{2}$ degrees of freedom remain.

Finally consider (A.1). We decompose the 6-index gauge field as

$$A_{\lambda_1 \dots \lambda_6} = \epsilon_{[\lambda_1}^i \dots \epsilon_{\lambda_6]}^{i_6} a_{i_1 \dots i_6} + \epsilon_{[\lambda_1}^i \dots \epsilon_{\lambda_5]}^{i_5} \bar{p}_{\lambda_6]} b_{i_1 \dots i_5}, \quad (\text{B.17})$$

where components along p_μ have already been set equal to zero. In momentum space (B.1) takes on the form

$$p^2 A_{\lambda_1 \dots \lambda_6} - p \cdot \bar{p} \epsilon_{[\lambda_1}^i \dots \epsilon_{\lambda_5]}^{i_5} p_{\lambda_6]} b_{i_1 \dots i_5} = 0. \quad (\text{B.18})$$

We conclude that the matrix $b_{i_1 \dots i_5}$ must vanish. The remaining degrees of freedom are massless and contained in the antisymmetric matrix $a_{i_1 \dots i_6}$, each index taking 8 values. Therefore, $A_{\lambda_1 \dots \lambda_6}$, restricted by (B.1), indeed describes 28 massless states.

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